COMPLEX NUMBERS

A complex number is represented by an expression of the form $a + b\boldsymbol{i}$ where a and b are real numbers and **i** is a symbol with a property $\boldsymbol{i^2} = -1$.

$\boldsymbol{i} = \sqrt{-1}$ was introduced by a Swiss mathematician Euler. Traditionally the letters Z and W are used to stand for complex numbers.

Given a complex numbers $z = a + b\boldsymbol{i}$.

The real part of a complex number z is $Re(z) = a$ and the imaginary part of z is $Im(z) = b$.

Both $Re(z)$ and $Im(z)$ real numbers.

Thus the real part of $\boldsymbol{Z} = 4 - 3\boldsymbol{i}$ is $Re(w) = 4$ and imaginary part of $\boldsymbol{Z}$ is $\mathrm{Im}(Z) = -3$

By identifying the real number $\boldsymbol{a}$ with a complex number $a + oi$ we consider ℝ (real numbers) to be subset of ℂ (complex numbers).

Consider the equation $x^2 + 9 = 0$, this can be written as $x^2 = -9$ and we can see that the equation has no real roots since we cannot find the real root of a negative number, But with $\boldsymbol{i}^2 = -1$ (Euler) we are able to find the square root of complex numbers.

$$x^2 = -9$$
$$x^2 = 9\boldsymbol{i^2}$$
$$\sqrt{\boldsymbol{x^2}} = \sqrt{9\,\boldsymbol{i}^2}$$
$$x = \pm 3\boldsymbol{i}$$
$$x = 3\boldsymbol{i} \qquad \boldsymbol{x} = -3\boldsymbol{i}$$

Example

Solve the following equations

(a) $4x^2 + 49 = 0$

(b) $x^2 + 2x + 6 = 0$

Solution

$$4x^2 + 49 = 0$$
$$4x^2 = -49$$
$$x^2 = -\frac{49}{4}$$
$$x^2 = \frac{49}{4}\boldsymbol{i}^2$$
$$x = \sqrt{\frac{49}{4}\boldsymbol{i}^2}$$
$$x = \pm\frac{7}{2}\boldsymbol{i}$$

(b) $x^2 + 2x + 6 = 0$

From, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$x = \frac{\left(-2 \pm \sqrt{(2)^2 - 4(1)(6)}\right)}{2 \times 1}$$
$$x = \frac{-2 \pm \sqrt{-20}}{2}$$
$$x = \frac{-2 \pm \sqrt{4\boldsymbol{i}^2 \times 5}}{2}$$
$$x = \frac{-2 \pm 2\boldsymbol{i}\sqrt{5}}{2}$$
$$x = -1 + \boldsymbol{i}\sqrt{5}$$
$$x = -1 - \boldsymbol{i}\sqrt{\boldsymbol{5}}$$

With this new concept we are in position to find the roots of any quadratic equation.

When the imaginary part of a complex number is zero, the complex number becomes a real number. Thus, all real numbers are complex numbers.

Definition

Given a complex number $\boldsymbol{z} = x + \boldsymbol{i}y$, the complex conjugate of **Z** denoted by $\bar{z}$ *or* z^* is a complex number given by $\bar{z} = x - \mathbf{i}y$. Therefore if $z = 4 + 3\boldsymbol{i}$, $w = -2 + 4\boldsymbol{i}$

Then $\bar{z} = 4 - 3\mathbf{i}$, $\bar{w}$ -2 – 4**i**

Algebra of complex numbers

1. Addition

Given that two complex numbers $z_1 = x_1 + \boldsymbol{i}y_1$, $z_2 = x_2 + \boldsymbol{i}y_2$. Then

$z_1 + z_2 = x_1 + \boldsymbol{i}y_1 + x_2 + \boldsymbol{i}y_2$

$= x_1 + x_2 + \boldsymbol{i}(y_1 + y_2)$

Therefore if $z_1 = 3 + 5\boldsymbol{i}$ and $z_2 = 2 - 7\boldsymbol{i}$

$$z_1 + z_2 = 3 + 5\boldsymbol{i} + 2 - 7\boldsymbol{i}$$
$$= (3 + 2) + 5\boldsymbol{i} - 7\boldsymbol{i}$$
$$= 5 - 2\boldsymbol{i}$$

Example

1. Subtraction:

$$z_1 = x_1 + \mathbf{i}y_1$$
$$z_2 = x_2 + \mathbf{i}y_2$$
$$z_1 - z_2 = (x_1 + \mathbf{i}y_1) - (x_2 + \mathbf{i}y_2)$$
$$= x_1 - x_2 + \mathbf{i}y_1 - iy_2$$
$$= (x_1 - x_2) + \mathbf{i}(y_1 - y_2)$$
$$z_1 = 4 - 3\boldsymbol{i}$$
$$z_2 = 6 - 14\boldsymbol{i}$$

Find $(z_1 - z_2)$

$$(z_1 - z_2) = (4 - 3\boldsymbol{i}) - (6 - 14\boldsymbol{i})$$
$$= 4 - 6 - 3\boldsymbol{i} + 14\boldsymbol{i}$$
$$= -2 + 11\boldsymbol{i}$$

2. Multiplication

$$z_1 = x_{\mathbf{1}} + \boldsymbol{i}y_{\mathbf{1}}, \quad z_2 = x_2 + \boldsymbol{i}y_2$$
$$z_1 z_2 = (x_1 + \boldsymbol{i}y_1)(x_2 + \boldsymbol{i}y_2)$$
$$= x_1x_2 + x_1y_2\boldsymbol{i} + y_1x_2\boldsymbol{i} + i^{\mathbf{2}}y_1y_2$$
$$= x_1x_2 - y_1y_2 + (y_1x_2 + x_1y_2)\boldsymbol{i}$$

Example

$z_1 = 3 + 5\boldsymbol{i}, \quad z_2 = 2 - 7\boldsymbol{i}$

Find z_1z_2

Solution

Find $z_1z_2 = (3 + 5i)(2 - 7i)$

$$= 3(2 - 7i) + 5i(2 - 7i)$$
$$= 6 - 21i + 10i - 35i^2$$
$$= 6 + 35 + (10 - 21)i$$
$$= 41 - 11i$$

3. Division

$$z_1 = x_1 + \boldsymbol{i}y_1 \text{ } and \text{ } z_2 = x_2 + iy_2$$
$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2}$$
$$\frac{z_1}{z_2} = \frac{x_1 + iy_1(x_2 - iy_2)}{x_2 + iy_2(x_2 - iy_2)}$$
$$\frac{z_1}{z_2} = \frac{x_1x_2 - x_1y_2i + x_2y_1i - \boldsymbol{i}^2y_1y_2}{(x_2)^{\mathbf{2}} - \boldsymbol{i}^{\mathbf{2}}{y_2}^{\mathbf{2}}}$$
$$= \frac{x_1x_2 + y_1y_2 + (x_2y_1 - x_1y_2)i}{{x_2}^2 + {y_2}^2}$$
$$= \frac{x_1x_2 + y_1y_2}{{x_2}^2 + {y_2}^2} + \frac{(x_2y_1 - x_1y_2)i}{{x_2}^2 + {y_2}^2}$$

Example I

Simplify $z = \frac{2+6i}{3-i}$

$$z = \frac{2 + 6i}{3 - i} = \frac{2 + 6i(3 + i)}{(3 - i)(3 + i)}$$
$$= \frac{2(3 + i) + 6i(3 + i)}{3^2 - i^2}$$
$$= \frac{6 + 2i + 18i + 6i^2}{10}$$
$$= \frac{6 + 20i - 6}{10}$$
$$= \frac{0 + 20i}{10}$$
$$= 2i$$

Example II

Express $\frac{-1+2i}{1+3i}$ in the form *a* + *b***i**

Solution

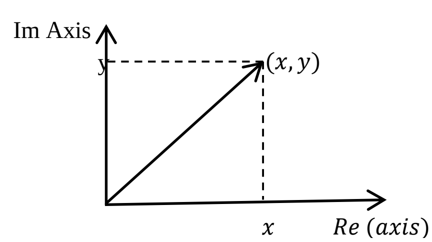
$$\frac{-1 + 2i}{1 + 3i} = \frac{-1 + 2i(1 - 3i)}{1 + 3i(1 - 3i)}$$
$$= \frac{-1 + 3i + 2i - 6i^2}{(1)^2 - (3i)^2}$$
$$= \frac{5i + 5}{10}$$
$$= \frac{5 + 5i}{10}$$
$$= \frac{1}{2} + \frac{1}{2}i$$

The Argand Diagram

Complex numbers can be represented graphically on a graph of Real (Re) and Imaginary (Im) axes called a **complex plane**. The complex plane is similar to the Cartesian plane where the imaginary axis corresponds to the *y*-axis and the real axis corresponds to the x-axis. The diagram representing the complex number in complex plane is called an **argand diagram** named after JR argand 1806.

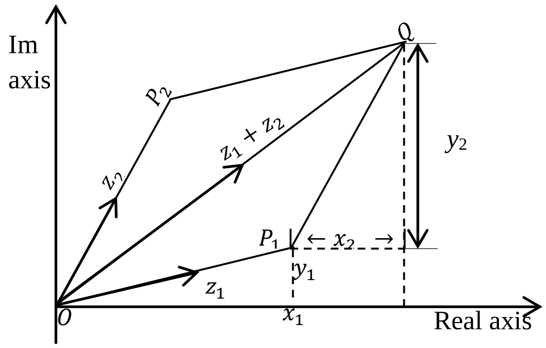
On the argand diagram a complex number is represented by a line with an arrow on the head to show direction

If $z = x + iy$ we can represent z on argand diagram as shown below.



If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ then

$$z_1 + z_2 = x_1 + x_2 + i(y_1 + y_2)$$

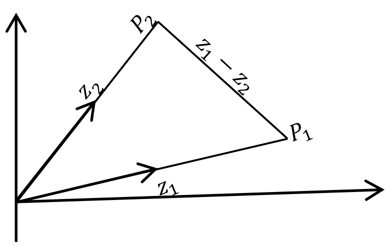


z_1, z_2 *and* $z_1 + z_2$ is represented by vectors $\overrightarrow{OP_1}$, $\overrightarrow{OP_2}$ *and* $\overrightarrow{OQ}$ respectively. The diagram shows that $\overrightarrow{P_1Q}$ is equal $\overrightarrow{OP_2}$ in magnitude and direction

$$\overrightarrow{OQ} = \overrightarrow{OP_1} + \overrightarrow{P_1Q} = \overrightarrow{OP_1} + \overrightarrow{OP_2}$$

Thus the sum of two complex numbers $z_1$and z_2 is represented in the argand diagram by the sum of the corresponding vectors $\overrightarrow{OP_1}$ *and* $\overrightarrow{OP_2}$

Representing $z_1 - z_2$ on the argand diagram.



$(z_1 - z_2) = OP_1 - OP_2$

$= \overrightarrow{P_1P_2}$

Since $\overrightarrow{OP_1} - \overrightarrow{OP_2} = \overrightarrow{P_1P_2}$

$z_1 - z_2$ can be represented by P_1P_2

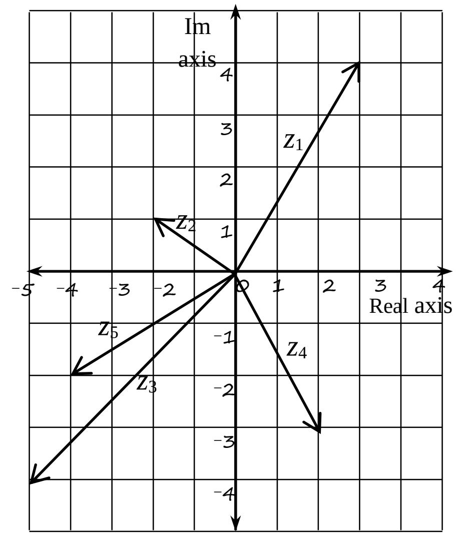
Example

Represent the following complex numbers on the argand diagram.

$z_1 = 3 + 4i, \quad z_2 = -2 + i, z_3 = -5 - 4i,$

$z_4 = 2 - 3i, \qquad z_5 = -4 - 2i,$

Solution



Modulus of a complex number

Given a complex number $z = x + iy$, the magnitude or length of z is denoted be $|z|$ is defined by

$$|z| = \sqrt{x^2 + y^2}$$

Example I

Given $z = 1 + \sqrt{3}i$ find $|z|$

Solution

$z = 1 + (\sqrt{3})i$

$|z| = \sqrt{(1)^2 + (\sqrt{3})^2}$

$= \sqrt{4}$

$= 2$

Example II

Find $|z|$ *if* $z = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$

Solution

$z = -\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}i$

$|z| = \sqrt{(\frac{1}{2})^2 + (\frac{-\sqrt{3}}{2})^2}$

$= \sqrt{\dfrac{1}{4} + \dfrac{3}{4}}$

$= \sqrt{1}$

$= 1$

$\Longrightarrow |z| = 1$

Example III

$z = -3 + 4i$ find $|z|$

Solution

$z = -3 + 4i$

$|z| = \sqrt{(-3)^2 + (4)^2}$

$= \sqrt{9 + 16}$

$= \sqrt{25}$

$= 5$

Properties of modulus

If z_1 *and* z_2 are complex numbers then

$(i)\ |z_1 z_2| = |z_1||z_2|$

$(ii)\ \left|\frac{z_1}{z_1}\right| = \frac{|z_1|}{|z_2|}$

Example I

$z_1 = 5 - 12i$ and $z_2 = 3 - 4i$

Find $|z_1 z_2|$ *and* $\left|\frac{z_1}{z_2}\right|$

Solution

$z_1 = 5 - 12i$, $\quad z_2 = 3 - 4i$

$$|z_1 z_2| = |z_1||z_2|$$

$$\Rightarrow |(5 - 12i)(3 - 4i)|$$

$$= |5 - 12i||3 - 4i|$$

$$= \sqrt{(5)^2 + (-12)^2}\sqrt{(3)^2 + (-4)^2}$$

$$= \sqrt{169} \times \sqrt{25}$$

$$= 13 \times 5$$

$$= 65$$

Alternatively

$$z_1 z_2 = (5 - 12i)(3 - 4i)$$

$$= 15 - 20i - 36i + 48i^2$$

$$= 15 - 48 - 56i$$

$$= -33 - 56i$$

$$|z_1 z_2| = \sqrt{(-33)^2 + (-56)^2}$$

$$= 65\ units$$

$$z_1 = 5 - 12i, \quad z_2 = 3 - 4i$$

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} = \frac{\sqrt{(5)^2 + (-12)^2}}{\sqrt{(3)^2 + (-4)^2}}$$

$$= \frac{13}{5}$$

Alternatively, $\frac{z_1}{z_2} = \frac{5 - 12i}{3 - 4i}$

$$\frac{(5 - 12i)(3 + 4i)}{(3 - 4i)(3 + 4i)}$$

$$\frac{z_1}{z_2} = \frac{15 + 20i - 36i - 48i^2}{(3)^2 - (4i)^2}$$

$$= \frac{63 - 16\mathrm{i}}{9 + 16}$$

$$= \frac{63}{25} - \frac{16}{25}i$$

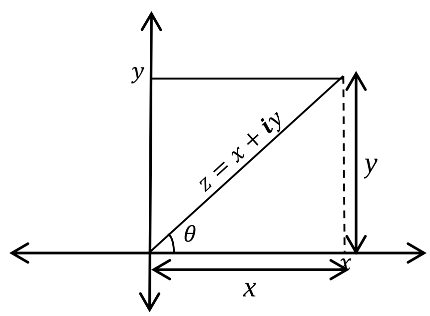
$$\left|\frac{z_1}{z_2}\right| = \sqrt{\left(\frac{63}{25}\right)^2 + \left(-\frac{16}{25}\right)^2}$$

$$\left|\frac{z_1}{z_2}\right| = \sqrt{\frac{(63)^2 + (16)^2}{25^2}}$$

$$= \frac{65}{25} = \frac{13}{5}$$

Argument of a complex number *Z* (arg Z)

The argument of a complex number z is defined to be the angle (θ) which the complex number *z* makes with the positive *x*-axis.



From the diagram above,

$$\tan\theta = \frac{y}{x} \Rightarrow \theta = tan^{-1}\left(\frac{y}{x}\right)$$

Note: For a given complex number, there will be infinitely many possible values of the argument, any two of which will differ by a whole multiple of 360°.

To avoid confusion we usually work with the value of θ for which $-\pi < \theta < \pi$ or $-180 < \theta < 180$. This is called the principle argument of z denoted by **arg z**.

In practice the formula $\tan\theta = \frac{y}{x}$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Which is often used to find the principal argument of a complex number z, despite the fact that it tends to two possible values for θ in the permitted range. The formula is necessary but not sufficient to help us obtain the arg z. The correct value of arg z is chosen with the aid of a sketch.

Example

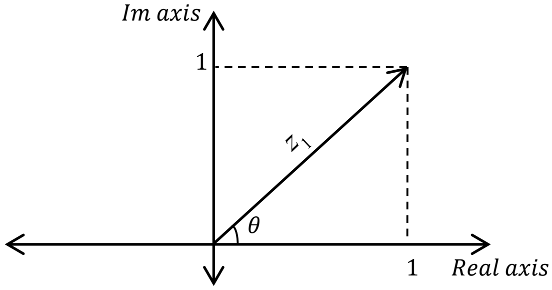
Find the principal argument of the following complex number

$(a)\ 1+i \quad (b) -1-i\sqrt{3} \quad (c) -5$

$(d) -\sqrt{3}+i \quad (e)\sqrt{3}-i$

Solution

Consider $z_1 = 1 + i$



$$\theta = tan^{-1}\left(\frac{1}{1}\right) = 45°$$

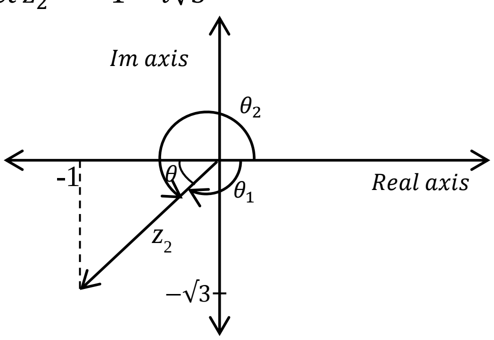
Since $180° = \pi$ radians

$$\theta = \frac{45\pi}{180} = \frac{\pi}{4}$$

$$\Longrightarrow \arg z_1 = 45°$$

$$\arg z_1 = \frac{45\pi}{180} = \frac{\pi}{4}$$

(b) Let $z_2 = -1 - i\sqrt{3}$



$$\tan\theta = \frac{\sqrt{3}}{1}$$

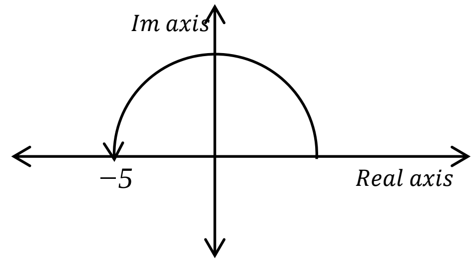
$$\theta = tan^{-1}\left(\frac{\sqrt{3}}{1}\right)$$

$$\theta = 60°$$

$$\arg z_2 = \theta_1$$

$$\Longrightarrow \arg z_2 = -120°$$

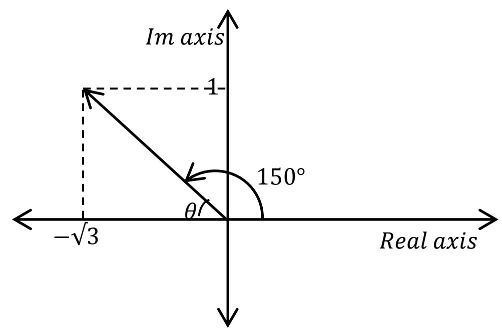
OR $\arg z_2 = -\frac{2}{3}\pi$



$$z_3 = -5 + 0i$$

$\arg z_3 = 180°$ or $\arg z_3 = \pi$

(d) Let $z_4 = -\sqrt{3} + i$



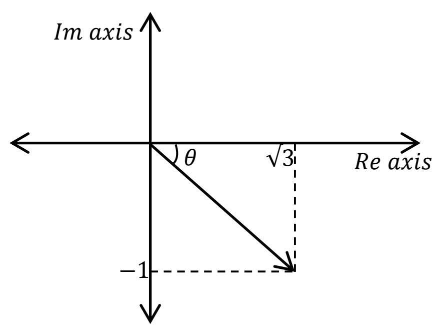
$$\theta = tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30°$$

$z_4 = -\sqrt{3} + i$

$\arg z_4 = 150°$, from the sketch above

(e) $\sqrt{3} - i$

$z_5 = \sqrt{3} - i$



$$\tan\theta = \frac{1}{\sqrt{3}}$$

$$\theta = tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30°$$

$\arg z_5 = -30°$ from the above diagram

Properties of Arguments

Given the two complex numbers z_1 and z_2 then

$$\mathbf{arg}(\boldsymbol{z_1 z_1}) = \mathbf{arg}\,\boldsymbol{z_1} + \mathbf{arg}\,\boldsymbol{z_2}$$

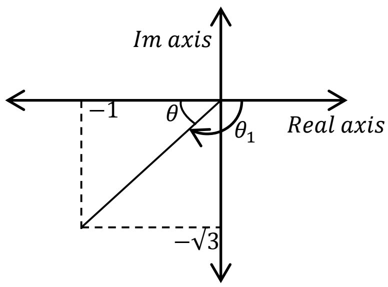
$$\mathbf{arg}\left(\frac{\boldsymbol{z_1}}{\boldsymbol{z_2}}\right) = \mathbf{arg}\,\boldsymbol{z_1} - \mathbf{arg}\,\boldsymbol{z_2}$$

Example I

Given that $z_1 = -1 - i\sqrt{3}$ and $z_2 = 1 + i$. Find the $\arg(z_1 z_2)$ *and* $\arg\left(\frac{z_1}{z_2}\right)$

Solution

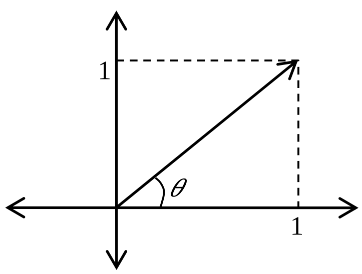
$$z_1 = -1 - i\sqrt{3}$$



$$\theta = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = 60°$$

$$\arg z_1 = \theta_1 = -120°$$

$$Z_2 = 1 + \mathbf{i}$$



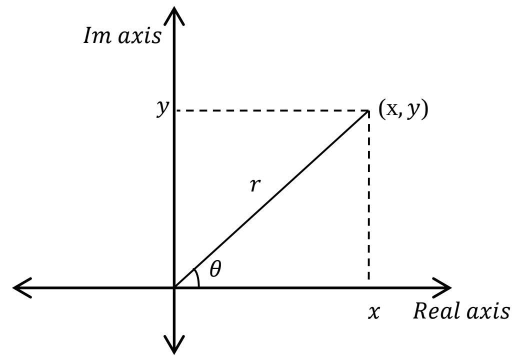
$$\arg z_2 = tan^{-1}\left(\frac{1}{1}\right) = 45°$$

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2$$
$$= -120 + 45°$$
$$= -75°$$

$$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$$
$$= -120 - 45$$
$$= -165$$

Modulus–argument form of a complex number

(*Polar form of a complex number*)



Consider a complex number $z = x + iy$ making an angle θ with the positive $x -$ axis

$$\arg z = \theta$$

From the diagram above $\cos\theta = \frac{x}{r}$ $\sin\theta = \frac{y}{r}$

$$x = r\cos\theta \quad r\sin\theta = y$$
$$z = x + iy$$
$$z = r\cos\theta + ir\sin\theta$$
$$z = r\ (\cos\theta + i\sin\theta)$$

(modulus argument form a complex number)

Where $r = |z| = \sqrt{x^2 + y^2}$

Example

Express the following complex numbers in modulus –argument

a) $5 + 5i\sqrt{3}$
b) $\sqrt{2} + i$
c) $-\frac{\sqrt{3}}{2} + \frac{1}{2}i$
d) $-3\sqrt{2} + 3\sqrt{2}i$
e) $-5i$
f) $-5 - 12i$

Solutions

$z_1 = 5 + 5i\sqrt{3}$

$r = \sqrt{(5)^2 + \left(5\sqrt{3}\right)^2}$

$= \sqrt{25 + 75}$

$= 10$

$\arg z_1 = tan^{-1}\left(\frac{5\sqrt{3}}{5}\right) = 60°$

$z_1 = 5 + 5i\sqrt{3} = 10(\cos 60 + i\sin 60)$

(b) $z_2 = \sqrt{2} + i$

$|z_2| = r = \sqrt{\left(\sqrt{2}\right)^2 + (1)^2}$

$= \sqrt{3}$

$\arg z_2 = tan^{-1}\left(\frac{1}{\sqrt{2}}\right) = 35.3°$

$z_2 = r(\cos\theta + i\sin\theta$

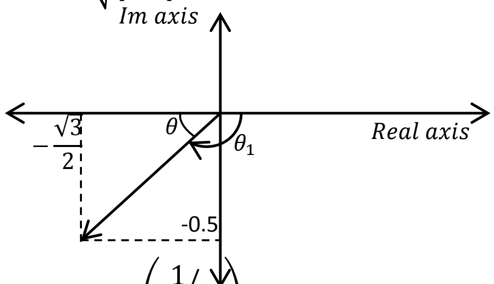
$\sqrt{3}[\cos 35.3 + i\sin 35.3]$

(c) $-\frac{\sqrt{3}}{2} - \frac{1}{2}i$

$z_3 = -\frac{\sqrt{3}}{2} - \frac{1}{2}i$

$|z_3| = \sqrt{\left(\left(-\frac{\sqrt{3}}{2}\right)^2 + \left(-\frac{1}{2}\right)^2\right)}$

$= \sqrt{\frac{3}{4} + \frac{1}{4}} = 1$



$\theta = tan^{-1}\left(\frac{1/2}{\sqrt{3}/2}\right) = 30°$

$z_3 = \frac{-\sqrt{3}}{2} - \frac{1}{2}\pi$

$\arg z_3 = -150°$

$z_3 = r(\cos\theta + i\sin\theta)$

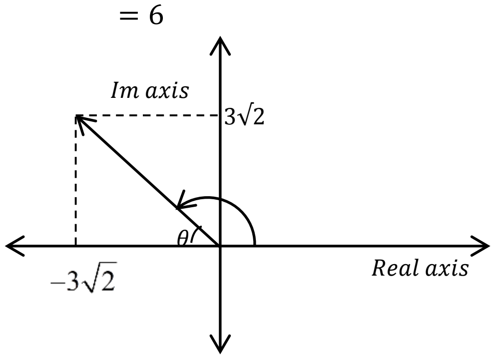
$z_3 = 1(\cos -150 + i\sin -150)$

(d) $z_4 = -3\sqrt{2} + (3\sqrt{2})$i

$|z_4| = \sqrt{\left(-3\sqrt{2}\right)^2 + \left(3\sqrt{2}\right)^2}$

$= \sqrt{36}$

$= 6$



$\theta = tan^{-1}\left(\frac{3\sqrt{2}}{3\sqrt{2}}\right)$

$\theta = 45°$

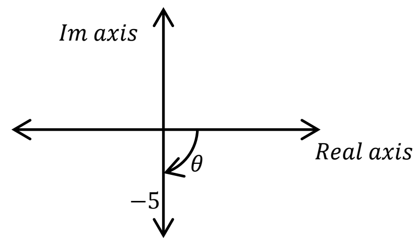
$\arg(z_4) = +135°$

$z_4 = 6(\cos 135 + i\sin 135)$

(e) $z_5 = -5i = 0 + -5i$

$|z_5| = \sqrt{0^2 + (-5)^2}$

$|z_5| = 5$



$\arg z_5 = -90$

$z_5 = 5(\cos -90 + i\sin -90)$

$z_6 = 3 + 4i$

$r = |z_6| = \sqrt{(3)^2 + (4)^2}$

$= \sqrt{25}$

$= 5$

$\theta = tan^{-1}\left(\frac{4}{5}\right) = 53.1$

$z_6 = 5(\cos 53.1° + i \sin 53.1°)$

(g) $z_6 = -5 - 12i$

$|z_6| = \sqrt{(-5)^2 + (-12)^2}$

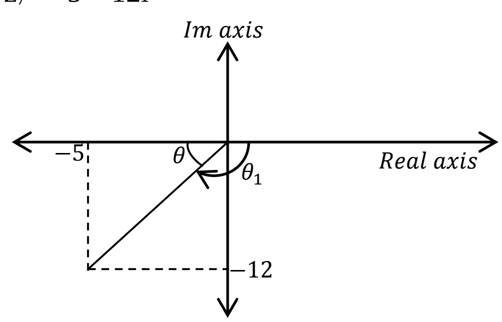
$= \sqrt{169}$

$= 13$

$\theta = tan^{-1}\left(\frac{12}{5}\right)$

$\theta = 67.4$

$z_7 = -5 - 12i$



$\arg z_7 = -112.6°$

$|z_7| = \sqrt{(-5)^2 + (-12)^2}$

$= \sqrt{25 + 144}$

$= \sqrt{169}$

$= 13$

$13(\cos -112.6 + i\sin -122.6)$

Example II

$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$
$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

Show that

$$z_1 z_2 = r_1 r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$$

And $\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2))$

Solution

$$z_1 z_2 = r_1(\cos\theta_1 + i\sin\theta_1)r_2(\cos\theta_2 + i\sin\theta_2)$$
$$= r_1 r_2[(\cos\theta_1\cos\theta_2 + \cos\theta_1\,(i\sin\theta_2) + i\sin\theta_1\cos\theta_2 + i^2\sin\theta_1\sin\theta_2)]$$
$$= r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)]$$
$$= r_1 r_2[(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))]$$

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\cos\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}$$
$$= \frac{r_1(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)}{r_1(\cos\theta_2 + i\sin\theta_2)(\cos\theta_2 - i\sin\theta_2)}$$
$$= \frac{r_1}{r_2}\left[\frac{\cos\theta_1\cos\theta_2 - i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2}{\cos^2\theta_2 + \sin^2\theta_2}\right]$$
$$= \frac{r_1}{r_2}\left(\frac{\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)}{cos^2\theta_2 + sin^2\theta_2}\right)$$
$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\frac{(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)}{1}$$
$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)$$

(as required)

Example III

Given that $z_1 = 1 + i$

$$z_2 = \sqrt{3} - i$$

Find in polar form $z_1 z_2$ and $\frac{z_1}{z_2}$

Solution

$$z_1 = 1 + i$$
$$|z_1| = \sqrt{1^2 + 1^2} = \sqrt{2}$$
$$\arg z_1 = tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$$
$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$$
$$z_1 = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$
$$z_2 = \sqrt{3} - i$$
$$|z_1| = r_2 = \sqrt{\left(\sqrt{3}\right)^2 + (-1)^2}$$
$$= \sqrt{4}$$
$$= 2$$
$$\arg z_2 = -30°$$
$$= -\frac{\pi}{6}\ radians$$
$$\arg z_2 = -\frac{\pi}{6}$$
$$z_2 = 2\left(\cos -\frac{\pi}{6} + i\sin -\frac{\pi}{6}\right)$$
$$z_1 z_2 = r_1 r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2))$$
$$= 2\sqrt{2}(\cos\left(\frac{\pi}{4} + -\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{4} + \frac{-\pi}{6}\right)$$
$$= 2\sqrt{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$$
$$\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2))$$
$$= \frac{\sqrt{2}}{2}\left[\cos\left(\frac{\pi}{4} - \frac{-\pi}{6}\right) + i\sin\left(\frac{\pi}{4} - \frac{-\pi}{6}\right)\right]$$
$$\frac{\sqrt{2}}{2}\left[\cos\left(\frac{5\pi}{12}\right) + i\sin\left(\frac{5\pi}{12}\right)\right]$$

Demoivre’s Theorem

Demoivres theorem states that for real values of n

$$(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$$

Proving Demoivre’s theorem by mathematical induction

$$(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$$

For *n* =1, $(\cos\theta + i\sin\theta)^1 = (\cos\theta + i\sin\theta)$

It’s true for *n* =1

Assume the results holds for the general value of *n*=*k*

$$(\cos\theta + i\sin\theta)^k = (\cos k\theta + i\sin k\theta)$$

It must be true for the next integer $n = k + 1$

$$(\cos\theta + i\sin\theta)^{k+1} = (\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta)$$
$$= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$$
$$= \cos k\theta\cos\theta + i\cos k\theta\sin\theta + i\sin k\theta\cos\theta + i^2\sin\theta\sin k\theta$$
$$= [(\cos k\theta\cos\theta - \sin k\theta\sin\theta) + i(\sin k\theta\cos\theta + \cos k\theta\sin\theta)]$$
$$= \cos(k\theta + \theta) + i\sin(k\theta + \theta)$$
$$= \cos(k+1)\theta + i\sin(k+1)\theta$$

$\Longrightarrow (\cos\theta + i\sin\theta)^{k+1}$

$$= \cos(k+1)\theta + i\sin(k+1)\theta$$

For the next integer , $n = k + 1 = 2$

$\Longrightarrow k = 1$

$$(\cos\theta + i\sin\theta)^2 = (\cos 2\theta + i\sin 2\theta)$$

Since it's true for n=1, n = 2 and so on it's true for all positive integral values of n.

Example I

Find the value of $\left(\cos\frac{1}{4}\pi + i\sin\frac{1}{4}\pi\right)^{12}$

Solution

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

$$\left(\cos\frac{1}{4}\pi + i\sin\frac{1}{4}\pi\right)^{12} =$$

$$\left(\cos\frac{1}{4}\pi \times 12 + i\sin\frac{1}{4}\pi \times 12\right)$$

$$= \cos 3\pi + i\sin 3\pi$$

$$= -1$$

Example II

Express $\left(1 - i\sqrt{3}\right)^4$ in the form $a + bi$

Solution

$\left(1 - i\sqrt{3}\right)^4$

Let $z = 1 - i\sqrt{3}$

$$|z| = \sqrt{(1)^2 + \left(-\sqrt{3}\right)^2} = 2$$

$$\arg z = -60$$

$$= -\frac{\pi}{3}$$

$$\arg z = -\frac{\pi}{3}$$

$$z = r(\cos\theta + i\sin\theta)$$

$$z = 2\left(\cos -\frac{\pi}{3} + i\sin -\frac{\pi}{3})\right)$$

$$z^4 = 2^4\left(\cos -\frac{\pi}{3} + i\sin -\frac{\pi}{3}\right)^4$$

$$= 16\left(\cos\tfrac{-4\pi}{3} + i\sin\tfrac{-4\pi}{3}\right)$$

$$= 16\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$$

$$= -8 + 8\sqrt{3i}$$

Example III

Evaluate $\frac{1}{\left(1 - i\sqrt{3}\right)^3}$

Solution:

$$\frac{1}{1 - i\sqrt{3}} = \left(1 - i\sqrt{3}\right)^{-3}$$

Let $z = \left(1 - i\sqrt{3}\right)$

$$|z| = \sqrt{1^2 + \left(-\sqrt{3}\right)^2}$$

$$= 2$$

$\arg z = -\frac{\pi}{3}$

$\left(1 - i\sqrt{3}\right) = 2\left(\cos -\frac{\pi}{3} + i\sin -\frac{\pi}{3}\right)$

$\left(1 - i\sqrt{3}\right)^{-3} = 2^{-3}\left(\cos -\frac{\pi}{3} + i\sin -\frac{\pi}{3}\right)^{-3}$

$$= \frac{1}{8}\left(\cos -\frac{\pi}{3} \times -3 + i\sin -\frac{\pi}{3} \times -3\right)$$

$$= \frac{1}{8}(\cos +\pi + i\sin +\pi)$$

$$= -\frac{1}{8}$$

Example IV

Express $\sqrt{3} + i$ in modulus –argument form. Hence find

$(\sqrt{3} + i)^{10}$ and $\frac{1}{\left(\sqrt{3} + i\right)^7}$ in the form of $a + bi$

Solution

Let $z = \sqrt{3} + i$

$$|z| = \sqrt{\left(\sqrt{3}\right)^2 + 1} = 2$$

$$\arg z = \frac{\pi}{6}$$

$$z = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$

$$\left(\sqrt{3} + i\right)^{10} = 2^{10}\left(\cos\left(\frac{10\pi}{6}\right) + i\sin\left(\frac{10\pi}{6}\right)\right)$$

$$= 2^{10}\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$$

$$= \frac{1024}{2} - \frac{i(1024)\sqrt{3}}{2}$$

$$= 512 - 512\sqrt{3}i$$

$$\frac{1}{\left(\sqrt{3} + i\right)^7} = \left(\sqrt{3} + i\right)^{-7}$$

$$= \left(2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\right)^{-7}$$

$$= 2^{-7}\left(\cos -7 \times \frac{\pi}{6} + i\sin -7 \times \frac{\pi}{6}\right)$$

$$= \frac{1}{128}\left(\cos\frac{-7\pi}{6} + i\sin\frac{-7\pi}{6}\right)$$
$$= \frac{1}{128}\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$$
$$= -\frac{\sqrt{3}}{256} + \frac{1}{256}i$$
$$= -\frac{\sqrt{3}}{256} + \frac{1}{256}i$$

Example V

Express (-1+i) in modulus – argument form. Hence show that $(-1+i)^{16}$ is real and that $\dfrac{1}{(-1+i)^6}$ is pure imaginary.

Solution

$z = -1 + i$

$|z| = \sqrt{(-1)^2 + 1^2}$

$= \sqrt{2}$

$\arg z = 135°$

$z = \sqrt{2}(\cos 135 + i\sin 135)$

$z^{16} = \left(\sqrt{2}\right)^{16}(\cos 135 \times 16 + i\sin 135 \times 16)$

$= 256(\cos 2160 + i\sin 2160)$

$= 256(1)$

$= 256$

$\Longrightarrow (-1+i)^{16} = 256$ So it is purely real

As required

$$\frac{1}{(-1+i)^6} = (-1+i)^{-6}$$

$z^{-6} = \left(\sqrt{2}\right)^{-6}(\cos 135 \times -6 + i\sin 135 \times -6)$

$$= \frac{1}{8}(0+i) = \frac{1}{8}i$$

$\Longrightarrow z^{-6}$ is purely imaginary.

Example VI

a) $(\cos\theta + i\sin\theta)^2(\cos\theta + i\sin\theta)^3$

b) $\dfrac{1}{(\cos\theta + i\sin\theta)^2}$

c) $\dfrac{\cos\theta + i\sin\theta}{(\cos\theta + i\sin\theta)^4}$

d) $\dfrac{\left(\cos\frac{\pi}{17} + i\sin\frac{\pi}{17}\right)^8}{\left(\cos\frac{\pi}{17} - i\sin\frac{\pi}{17}\right)^9}$

e) $\dfrac{(\cos\theta + i\sin\theta)(\cos 2\theta + i\sin 2\theta)}{\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)}$

f) $\dfrac{\left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)^8}{\left(\cos\frac{3\pi}{5} - i\sin\frac{3\pi}{5}\right)^3}$

Solutions

(a) $(\cos\theta + i\sin\theta)^2(\cos\theta + i\sin\theta)^3$

$= (\cos\theta + i\sin\theta)^{2+3}$

$= (\cos\theta + i\sin\theta)^5$

$= (\cos 5\theta + i\sin 5\theta)$

(b) $\dfrac{1}{(\cos\theta + i\sin\theta)^2}$

$= (\cos\theta + i\sin\theta)^{-2}$

$= \cos -2\theta + i\sin -2\theta$

$= \cos 2\theta - i\sin 2\theta$

(c) $\dfrac{\cos\theta + i\sin\theta}{(\cos\theta + i\sin\theta)^4}$

$= (\cos\theta + i\sin\theta)^1(\cos\theta + i\sin\theta)^{-4}$

$= (\cos\theta + i\sin\theta)^{1+-4}$

$= (\cos\theta + i\sin\theta)^{-3}$

$= \cos -3\theta + i\sin -3\theta$

$= (\cos 3\theta - i\sin 3\theta)$

(d) $\dfrac{(\cos\frac{\pi}{17} + i\sin\frac{\pi}{17})^8}{(\cos\frac{\pi}{17} - i\sin\frac{\pi}{17})^9}$

$$\frac{(\cos\pi + i\sin\pi)^{8/17}}{(\cos\pi + i\sin\pi)^{-9/17}}$$

$$(\cos\pi + i\sin\pi)^{\frac{8}{17}-\frac{-9}{17}}$$

$$(\cos\pi + i\sin\pi)^1$$

(e) $\dfrac{(\cos\theta + i\sin\theta)(\cos 2\theta + i\sin 2\theta)}{(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2})}$

$$\frac{(\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta)^2}{(\cos\theta + i\sin\theta)^{1/2}}$$

$$\frac{(\cos\theta + i\sin\theta)^3}{(\cos\theta + i\sin\theta)^{1/2}}$$

$$(\cos\theta + i\sin\theta)^{5/2}$$

$$\cos\tfrac{5}{2}\theta + i\sin\tfrac{5}{2}\theta$$

(f) $\dfrac{\left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)^8}{\left(\cos\frac{3\pi}{5} - i\sin\frac{3\pi}{5}\right)^3}$

$$\frac{\left(\left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)^2\right)^8}{\left(\left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)^{-3}\right)^3}$$

$$\frac{\left(\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}\right)^{16}}{\left(\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}\right)^{-9}}$$

$$\left(\cos\frac{\pi}{5}+i\sin\frac{\pi}{5}\right)^{16--9}$$

$$\cos\left(25\times\frac{\pi}{5}\right)+i\sin\left(25\times\frac{\pi}{5}\right)$$

$$=(\cos 5\pi+i\sin 5\pi)$$

Example VII

Use De-moivre's theorem to show that

$$\tan 3\theta=\frac{3\tan\theta-3\tan^3\theta}{1-3\tan^2\theta}$$

Solution

$$(\cos 3\theta+i\sin 3\theta)=(\cos\theta+i\sin\theta)^3$$

$$but\ (\cos\theta+i\sin\theta)^3=$$

$$=cos^3\theta+3(i\sin\theta)cos^2\theta+3(i\sin\theta)^2\cos\theta+(i\sin\theta)^3$$

$$=(\cos^3\theta-3\sin^2\theta\cos\theta)+i(3\sin\theta\cos^2\theta-\sin^3\theta)$$

$$=\cos 3\theta+i\sin 3\theta$$

Equating real to real and imaginary to imaginary;

$$\Rightarrow\sin 3\theta=3\sin\theta\, cos^2\theta-sin^3\theta\ \ldots..(1)$$

$$\cos 3\theta=\cos^3\theta-3sin^2\theta\cos\theta\ \ldots\ldots\ldots\ldots..\ldots(2)$$

Eqn (1) ÷ Eqn (2)

$$\Rightarrow\tan 3\theta=\frac{3\sin\theta\, cos^2\theta-sin^3\theta}{cos^3\theta-3sin^2\theta\cos\theta}$$

$$\tan 3\theta=\frac{\frac{3\sin\theta\, cos^2\theta}{\cos^3\theta}-\frac{\sin^3\theta}{cos^3\theta}}{\frac{\cos^3\theta}{\cos^3\theta}-\frac{3\sin^2\theta\cos\theta}{\cos^3\theta}}$$

$$\tan 3\theta=\frac{3\tan\theta-\tan^2\theta}{1-3\tan^2\theta}$$

Example VIII

Use Demovre's theorem to show that

$$\tan 4\theta=\frac{4\tan\theta-4\tan^3\theta}{1-6\tan^2\theta+\tan^4\theta}$$

Solution

$(\cos\theta+i\sin\theta)^4=(\cos\theta+i\sin\theta)^4$

$=\cos^4\theta+4\cos^3\theta\,(i\sin\theta)+6\cos^2\theta\,(i\sin\theta)^2+$
$4\cos\theta\,(i\sin\theta)^3+(i\sin\theta)^4$

$$=\cos^4\theta+(4\cos^3\theta\sin\theta)i-6\,cos^2\theta\, sin^2\theta-(4\cos\theta\sin^3\theta)\,i+\sin^4\theta$$

$$=(\cos 4\,\theta+i\sin 4\theta)$$

Equating real to real and imaginary to imaginary

$$\cos 4\theta=\cos^4\theta-6\,cos^2\theta sin^2\theta+\sin^4\theta\ \ldots.(i)$$

$$\sin 4\theta=4\cos^3\theta\sin\theta-4\cos\theta\sin^3\theta\ \ldots\ldots\ldots(ii)$$

Eqn (ii) ÷ Eqn (i)

$$\tan 4\theta=\frac{4\cos^3\theta\sin\theta-4\cos\theta\sin^3\theta}{\cos^4\theta-6\,cos^2\theta sin^2\theta+\sin^4\theta}$$

$$\tan 4\theta=\frac{\frac{4\cos^3\theta\sin\theta}{\cos^4\theta}-\frac{4\cos\theta\sin^3\theta}{\cos^4\theta}}{\frac{\cos^4\theta}{\cos^4\theta}-\frac{6cos^2\theta\, sin^2\theta}{\cos^4\theta}+\frac{\sin^4\theta}{\cos^4\theta}}$$

$$\tan 4\theta=\frac{4\tan\theta-4\tan^3\theta}{1-6tan^2\theta+\tan^4\theta}$$

Example IX

Show that

$$z^n+\frac{1}{z^n}=2\cos n\theta$$

$$z^n-\frac{1}{z^n}=2i\sin n\theta$$

Hence show that $\cos^4\theta=\frac{1}{8}(\cos 4\theta+4\cos 2\theta+3)$

Solution

$$z=\cos\theta+i\sin\theta$$

$$z^n=(\cos\theta+i\sin\theta)^n$$

$$=(\cos n\theta+i\sin n\theta)$$

$$z^{-n}=(\cos\theta+i\sin\theta)^{-n}$$

$$=\cos -n\theta+i\sin -n\theta$$

$$=\cos n\theta-i\sin n\theta$$

$$z^n+\frac{1}{z^n}=\cos n\theta+i\sin n\theta+\cos n\theta-i\sin n\theta$$

$$=2\cos n\theta$$

$$z^n-\frac{1}{z^n}$$

$$=(\cos n\theta+i\sin n\theta)-(\cos n\theta-i\sin n\theta)$$

$$=2i\sin n\theta$$

$$\text{from } z^n+\frac{1}{z^n}=2\cos n\theta$$

$$z+\frac{1}{z}=2\cos\theta$$

$$z^n-\frac{1}{z^n}=2i\sin n\theta$$

$$z-\frac{1}{z}=2i\sin\theta$$

$$z + \frac{1}{z} = 2\cos\theta$$

$$\left(z + \frac{1}{z}\right)^4 = (2\cos\theta)^4$$

But $(z + \frac{1}{z})^4$

$$z^4 + 4z^3\left(\frac{1}{z}\right) + 6z^2\left(\frac{1}{z}\right)^2 + 4z\left(\frac{1}{z}\right)^3 + \left(\frac{1}{z}\right)^4$$

$$\left(z^4 + \frac{1}{z^4}\right) + 4\left(z^2 + \frac{1}{z^2}\right) + 6 = \left(z + \frac{1}{z}\right)^4$$

$$2\cos 4\,\theta + 4(2\cos 2\theta) + 6 = (2\cos\theta)^4$$

$$16\cos^4\theta = 2\cos 4\theta + 4(2\cos 2\theta) + 6$$

$$\cos^4\theta = \frac{1}{16}(2\cos 4\theta + 8\cos 2\theta + 6)$$

$$\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$$

Example XI

Given that $z = \cos\theta + i\sin\theta$ show that

$$z^n - \frac{1}{z^n} = 2i\sin n\theta$$

Hence or otherwise show that

$$\sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$$

Solution

$$z^n - \frac{1}{z^n} = 2i\sin\theta$$

$$z - \frac{1}{z} = 2i\sin\theta$$

$$\left(z - \frac{1}{z}\right)^5 = (2i\sin\theta)^5$$

$$\left(z - \frac{1}{z}\right)^5 = i^5(32)\sin^5\theta$$

$$\left(z - \frac{1}{z}\right)^5 = (32i \times i^4\sin^5\theta)$$

$$\left(z - \frac{1}{z}\right)^5 = 32i\sin^5\theta$$

but $\left(z - \frac{1}{z}\right)^5 = z^5 + 5z^4\left(-\frac{1}{z}\right) + 10z^3\left(-\frac{1}{z}\right)^2$

$+10z^2\left(-\frac{1}{z}\right)^3 + 5z\left(-\frac{1}{z}\right)^4 + \left(-\frac{1}{z}\right)^5$

$$= z^5 - \frac{1}{z^5} - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right)$$

$$z^n - \frac{1}{z^n} = 2i\sin n\theta$$

$$z^5 - \frac{1}{z^5} = 2i\sin 5\theta$$

$$z - \frac{1}{z} = 2i\sin\theta$$

$$\left(z - \frac{1}{z}\right)^5 = (2i\sin\theta)^5$$

$$2i\sin 5\theta - 5(2i\sin 3\theta) + 10(2i\sin\theta) = 32i\sin^5\theta$$

$$\sin^5\theta = \frac{1}{32}(2\sin 5\theta - 10\sin 3\theta + 20\sin\theta)$$

$$\sin^5\theta = \frac{1}{16}(\sin 5\theta - 5\sin 3\theta + 10\sin\theta)$$

Example XII

Prove that $\cos^6\theta + \sin^6\theta = \frac{1}{8}(3\cos 4\theta + 5)$

Solution

$$z^n + \frac{1}{z^n} = 2\cos n\theta$$

$$z + \frac{1}{z} = 2\cos\theta$$

$$\left(z + \frac{1}{z}\right)^6 = (2\cos\theta)^6$$

$$\left(z + \frac{1}{z}\right)^6 = 64\cos^6\theta$$

But $\left(z + \frac{1}{z}\right)^6 = z^6 + 6z^5\left(\frac{1}{z}\right) + 15z^4\left(\frac{1}{z}\right)^2 +$

$$20z^3\left(\frac{1}{z}\right)^3 + 15z^2\left(\frac{1}{z}\right)^4 + 6z\left(\frac{1}{z}\right)^5 + \left(\frac{1}{z}\right)^6$$

$$= \left(z^6 + \frac{1}{z^6}\right) + \left(6z^4 + \frac{6}{z^4}\right) + \left(15z^2 + \frac{15}{z^2}\right) + 20$$

$$= 2\cos 6\theta + 6(2\cos 4\theta) + 15(2\cos 2\theta) + 20$$

$\Rightarrow 64\cos^6\theta = 2\cos 6\theta + 12\cos 4\,\theta + 30\cos 2\theta + 20$

$64\cos^6\theta = 2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta$

$+ 20 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. (1)$

$\left(z - \frac{1}{z}\right) = 2i\sin\theta$

$\left(z - \frac{1}{z}\right)^6 = 64i^6\sin^6\theta$

$\left(z - \frac{1}{z}\right)^6 = -64\sin^6\theta$

But

$\left(z - \frac{1}{z}\right)^6 = z^6 + 6z^5\left(-\frac{1}{z}\right) + 15z^4\left(-\frac{1}{z}\right)^2 +$

$20z^3\left(-\frac{1}{z}\right)^3 + 15z^2\left(-\frac{1}{z}\right)^4 + 6z\left(-\frac{1}{z}\right)^5 + \left(-\frac{1}{z}\right)^6$

$$= \left(z^6 + \frac{1}{z^6}\right) - 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) - 20$$

$$= 2\cos 6\theta - 6(2\cos 4\theta) + 15(2\cos 2\theta) - 20$$

$$\Rightarrow 2\cos 6\theta - 12\cos 4\theta + 30\cos 2\theta - 20 = -64\sin^6\theta$$

$$-64\sin^6\theta = 2\cos 6\theta - 12\cos 4\theta + 30\cos 2\theta - 20 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Eqn (2) – Eqn (1)

$$\Rightarrow 64\cos^6\theta - -64\sin^6\theta = 24\cos 4\theta + 40$$

$$\cos^6\theta + \sin^6\theta = \frac{8}{64}(3\cos 4\theta + 5)$$

$$\cos^6\theta + \sin^6\theta = \frac{1}{8}(3\cos 4\theta + 5)$$

Solving Complex Equations

Given that x and y are real numbers. Find the values of x and y which satisfy the equation.

$$\frac{2y+4i}{2x+y} - \frac{y}{x-i} = 0$$

Solution

$$\frac{2y+4i}{2x+y} - \frac{y}{x-i} = 0$$

$$\frac{2y+4i}{2x+y} = \frac{y}{x-i}$$

$$\frac{2y+4i}{2x+y} = \frac{y}{x-i} \times \frac{x+i}{x+i}$$

$$\frac{2y+4i}{2x+y} = \frac{xy+iy}{x^2+1}$$

$$\frac{2y}{2x+y} + \frac{4i}{2x+y} = \frac{xy}{x^2+1} + \frac{yi}{x^2+1}$$

Equating real to real and imaginary to imaginary

$$\Rightarrow \frac{2y}{2x+y} = \frac{xy}{x^2+1} \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$\frac{4}{2x+y} = \frac{y}{x^2+1} \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

From equation (1)

$$2y(x^2+1) = xy(2x+y)$$

$$2x^2y + 2y = 2x^2y + xy^2$$

$$2y - xy^2 = 0$$

$$y(2-xy) = 0$$

$$y = 0 \ \ or\ xy = 2$$

From Eqn (2), $\frac{4}{2x+y} = \frac{y}{x^2+1}$

$4x^2 + 4 = 2xy + y^2$

For $y = 0$, $4x^2 + 4 = 0$

$x^2 + 1 = 0$

$x^2 = -1$

$x^2 = i^2$

$x = \pm i$

For $xy = 2$, $y = \frac{2}{x}$

$$\Rightarrow 4x^2 + 4 = 4 + \frac{4}{x^2}$$

$$4x^2 - \frac{4}{x^2} = 0$$

Let $x^2 = m$

$$4m - \frac{4}{m} = 0$$

$4m^2 - 4 = 0$

$m^2 - 1 = 0$

$(m+1)(m-1) = 0$

$m = 1$, $m = -1$

When $m = 1$, $x^2 = 1 \Rightarrow x = \pm 1$

When $m = -1$, $x^2 = i^2 \Rightarrow x = \pm i$

$xy = 2$

If $x = 1$, $y = 2$

If $x = -1$, $y = -2$

If $x = i$, $y = -2i$

If $x = -i$, $y = 2i$

Example II

Find the values of x and y in

$$\frac{x}{2+3i} - \frac{y}{3-2i} = \frac{6+2i}{1+8i}$$

Solution

$$\frac{x}{2+3i} - \frac{y}{3-2i} = \frac{6+2i}{1+8i}$$

$$\frac{x(2-3i)}{2+3i(2-3i)} - \frac{y(3+2i)}{(3-2i)(3+2i)} = \frac{(6+2i)(1-8i)}{(1+8i)(1-8i)}$$

$$\frac{(2x-3xi)}{13} - \frac{(3y+2yi)}{13} = \frac{(6-48i+2i+16)}{65}$$

$$\frac{2x-3y}{13} - \frac{3x+2y}{13}i = \frac{22}{65} - \frac{46}{65}i$$

$$\Rightarrow \frac{2x-3y}{13} = \frac{22}{65}$$

$$5(2x-3y) = 22$$

$$10x - 15y = 22 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Similarly, $\frac{3x+2y}{13} = \frac{46}{65}$

$$5(3x+2y) = 46$$

$$15x + 10y = 46 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Solving eqn (1) and eqn (2) simultaneously

$$\Longrightarrow x = 2.8 \quad y = 0.4$$

Example III

Find the values of x and y if $\frac{x}{1+i} + \frac{y}{2-i} = 2 + 4i$

Solution

$$\frac{x}{1+i}+\frac{y}{2-i}=2+4i$$

$$\frac{x(1-i)}{(1+i)(1-i)}+\frac{y(2+i)}{(2-i)(2+i)}=2+4i$$

$$\frac{x-xi}{2}+\frac{2y+yi}{5}=2+4i$$

$$5(x-xi)+2(2y+yi)=2+4i$$

$$5x-5xi+4y+2yi=20+40i$$

Equating real to real and imaginary to imaginary;

$$5x+4y=20 \ldots\ldots\ldots\ldots\ldots (1)$$

$$2y-5x=40 \ldots\ldots\ldots\ldots\ldots (2)$$

Solving Eqn (1) and Eqn (2) simultaneously;

$$y=10$$

$$x=-4$$

Example IV

Find the values of x and y. given that

$$\frac{xi}{1+iy}=\frac{3x+4i}{x+3y}$$

Solution

$$\frac{xi(1-iy)}{(1+iy)(1-iy)}=\frac{(3x+4i)}{x+3y}$$

$$\frac{xi+xy}{1+y^2}=\frac{3x+4i}{x+3y}$$

$$\frac{xy}{1+y^2}+\frac{xi}{1+y^2}=\frac{3x}{x+3y}+\frac{4i}{x+3y}$$

$$\Rightarrow \frac{xy}{1+y^2}=\frac{3x}{x+3y} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots. (1)$$

$$\frac{x}{1+y^2}=\frac{4}{x+3y} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

From equation (1)

$$x^2y+3xy^2=3x+3xy^2$$

$$\Rightarrow x^2y=3x$$

$$\Rightarrow x^2y-3x=0$$

$$x(xy-3)=0$$

$$x=0 \quad or\; xy=3$$

From eqn (2)

$$x^2+3xy=4+4y^2 \ldots\ldots\ldots(3)$$

When $x = 0$, $0=4+4y^2$

$$-1=y^2$$

$$y=\pm i$$

When $xy = 3$

$$y=\frac{3}{x}$$

Substituting $y = \frac{3}{x}$ and $xy = 3$ in Eqn (3);

$$x^2+3(3)=4+4\left(\frac{3}{x}\right)^2$$

$$x^2+9=4+\frac{36}{x^2}$$

$$x^2+9=4+\frac{36}{x^2}$$

$$x^2-\frac{36}{x^2}+5=0$$

$$let\; x^2=P$$

$$P-\frac{36}{P}+5=0$$

$$P^2-36+5P=0$$

$$P^2+5P-36=0$$

$$(P+9)(P-4)=0$$

$$(x^2+9)(x^2-4)=0$$

$$x^2-4=0 \quad \Rightarrow \quad x=\pm 2$$

$$x=2, \quad x=-2$$

When $x=2, \quad y=\frac{3}{2}$

When $x=-2, \quad y=-\frac{3}{2}$

$$x^2+9=0$$

$$x^2=-9$$

$$x=\pm 3i$$

$$when\; x=3i$$

$$y=\frac{3}{3i}=\frac{1}{i}$$

$$y=-i$$

$$when\; x=-3i$$

$$y=+i$$

Example V

If z is a complex number such that $z=\frac{P}{2-i}+\frac{q}{1+3i}$. Where p and q are real. If |z| = 7, arg P = $\frac{\pi}{2}$. Find the value of p and q.

Solution

$$z=\frac{P}{2-i}+\frac{q}{1+3i}$$

$$\mathrm{z}=\frac{\mathrm{P}(2+\mathrm{i})}{(2-\mathrm{i})(2+\mathrm{i})}+\frac{q(1-3\mathrm{i})}{(1+3\mathrm{i})(1-3\mathrm{i})}$$

$$\mathrm{z}=\frac{2\mathrm{p}+\mathrm{Pi}}{5}+\frac{\mathrm{q}-3\mathrm{qi}}{10}$$

$$z=\frac{2(2p+pi)+q-3qi}{10}$$
$$z=\frac{4p+2pi+q-3qi}{10}$$
$$z=\frac{4p+q+(2p-3q)i}{10}$$
$$\arg z=\tan^{-1}\left(\frac{\frac{2P-3q}{10}}{\frac{4P+q}{10}}\right)=\frac{\pi}{2}$$
$$tan^{-1}(\frac{2p-3q}{4P+q})=\frac{\pi}{2}$$
$$\frac{2\mathrm{p}-3\mathrm{q}}{4\mathrm{P}+\mathrm{q}}=\infty$$
$$4\mathrm{P}+\mathrm{q}=0$$
$$\mathrm{q}=-4\mathrm{P}$$
$$|\mathrm{z}|=7$$
$$\sqrt{\left(\frac{4\mathrm{P}+\mathrm{q}}{10}\right)^2+\left(\frac{2P-3q}{10}\right)^2}=7 \ \ldots\ldots\ldots\ldots\ldots \ (1)$$
Substituting q = -4p in Eqn (1)
$$\sqrt{0^2+\left(\frac{14p}{10}\right)^2}=7$$
$$\frac{14p}{10}=7$$
$$p=5$$
$$\mathrm{q}=-4\times 5$$
$$q=-20$$

Example VI

Given that (1 + 5i)p – 2q = 3 + 7i, find p and q

(a) When p and q are real

(b) When p and q are conjugate complex numbers

Solution

(a) $(1+5i)P-2q=3+7i$

$P+5Pi-2q=3+7i$

$P-2q+5Pi=3+7i$

$P-2q=3$ …………………….. (1)

$5\mathrm{P}=7$ ………………………… (2)

From Eqn (2),

$$\mathrm{P}=\frac{7}{5}$$
$$\Rightarrow\frac{7}{5}-2q=3$$
$$\frac{7}{5}-3=2q$$
$$-\frac{8}{5}=2q$$
$$-\frac{8}{10}=q$$
$$q=-\frac{4}{5}$$
$$\Rightarrow p=\frac{7}{5},\quad q=-\frac{4}{5}$$

(b) Let $p=x+iy$

$q=x-iy$

$(1+5i)(x+iy)-2(x-iy)=3+7i$

$x+iy+5xi-5y-2x+2yi=3+7i$

$(x-5y-2x)+(y+5x+2y)i=3+7i$

$(-x-5y)+(3y+5x)i=3+7i$

$-x-5y=3$

$x=-3-5y$ …………………… (1)

$3y+5x=7$ …………………….. (2)

Substituting Eqn (1) in Eqn (2)

$3y+5(-3-5y)=7$

$3\mathrm{y}-15-25\mathrm{y}=7$

$-22\mathrm{y}=22$

$y=-1$

$x=-3-5(-1)$

$x=-3+5$

$x=2$

$p=x+iy$

$p=2-i$

$q=2+i$

Square root of Complex Numbers

Example I

Find the square root of $35-12i$

Solution

Let $\sqrt{35-12i}=x+iy$

$\left(\sqrt{35-12i}\right)^2=(x+iy)^2$

$35-12i=x^2+2xyi+i^2y^2$

$35-12i=x^2-y^2+2xyi$

$\Rightarrow x^2-y^2=35$

$2xy=-12$

$xy=-6$

$y=-\frac{6}{x}$

$x^2-\frac{36}{x^2}=35$

$x^4-36=35x^2$

$x^4-35x^2-36=0$

Let $x^2=m$

$\mathrm{m}^2-35m-36=0$

$$(m-36)(m+1)=0$$
$$(x^2-36)(x^2+1)=0$$

But x is real

$\Rightarrow x^2-36=0$

$x=\pm 6$

When $x=6$ $\quad y=-\frac{6}{6}$

$y=-1$

$when\ x=-6, \quad y=1$

$\Rightarrow \sqrt{35-12i}=6-i$

or $\sqrt{35-12i}=-6+i$

Example VIII

Find the square root of $5-12i$

solution

Let $\sqrt{5-12i}=x+iy$

$$5-12i=(x+iy)^2$$
$$5-12i=x^2+2xyi+yi^2$$
$$5-12i=x^2-y^2+2xyi$$

Equating real to real and imaginary to imaginary;

$$\Rightarrow x^2-y^2=5 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$
$$2xy=-12$$
$$xy=-6$$
$$y=-\frac{6}{x} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Substituting Eqn (2) in Eqn (1)

$$x^2-\frac{36}{x^2}=5$$
$$(x^2)^2-36=5x^2$$
$$let\ m=x^2$$
$$m^2-36=5m$$
$$m^2-5m-36=0$$
$$(m-9)(m+4)=0$$
$$(x^2-9)(x^2+4)=0$$
$$x^2=9$$
$$x=\pm 3$$
$$when\ x=3, \ y=-2$$
$$when\ x=-3, \quad y=2$$
$$\sqrt{5-12i}=3-2i$$
$$or\ \sqrt{5-12i}=3+2i$$

Example IX

Find the roots of $z^2-(1-i)z+7i-4=0$

Solution

$$z=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$z=\frac{(1-i)\pm\sqrt{(1-i)^2-4(1)(7i-4)}}{2\times 1}$$

$$z=\frac{1-i\pm\sqrt{1-2i-1-28i+16}}{2}$$

$$\frac{1-i\pm\sqrt{16-30i}}{2}$$

But $\sqrt{16-30i}=a+bi$

$16-30i=a^2+2abi-b^2$

$a^2-b^2=16$

$2ab=-30$

$ab=-15$

$a=\frac{-15}{b}$

$\left(\frac{15}{b}\right)^2-b^2=16$

$\frac{225}{b^2}-b^2=16$

Let $m=b^2$

$\frac{225}{m}-m=16$

$m^2+16m-225=0$

$m=9, \quad m=-25$

$b^2=9$

$b=\pm 3$

$ab=-15$

$a=5$

When $b=-3, \quad a=5$

When $b=3, \quad a=-5$

$$a+bi=5-3i,\ -5+3i$$
$$\sqrt{16-30i}=\pm(5-3i)$$
$$z=\frac{1-i\pm(5-3i)}{2}$$
$$z=3-2i$$
$$z=-2+i$$

Example X

Show that $1+2i$ is a root of the equation

$$2z^3-z^2+4z+15=0$$

Solution

$$z=1+2i$$
$$z^2=(1+2i)^2$$
$$=1+4i+4i^2$$
$$=-3+4i$$
$$z^3=z\times z^2=(1+2i)(-3+4i)$$
$$=-3+4i-6i-8$$
$$=-11-2i$$

$2z^3 - z^2 + 4z + 15$

$\Rightarrow 2(-11-2i) - (-3+4i) + (4(1+2i) + 15$

$= -22 - 4i + 3 - 4i + 4 + 8i + 15$

$= -22 + 22 - 8i + 8i$

$= 0 + 0i$

$= 0$

$\Rightarrow 1 + 2i$ is a root of the equation.

Since $z = 1 + 2i$ is a root of the equation

$$2z^3 - z^2 + 4z + 15 = 0$$

The complex conjugate $\bar{z} = 1 - 2$i must also be a root of the above equation

$\Rightarrow 1 - 2i = z$ is also a root of the equation

$$2z^3 - z^2 + 4z + 15 = 0$$
$$2z^3 - z^2 + 4z + 15 = 0$$
$$z = 1 + 2i$$
$$z = 1 - 2i$$

Sum of roots $= 1 + 2i + 1 - 2i$

$= 2$

Product of roots = $(1)^2 - (2i)^2$

$= 1 + 4$

$= 5$

$$z^2 - \begin{pmatrix} sum\ of \\ roots \end{pmatrix} z + product = 0$$
$$z^2 - 2z + 5 = 0$$

$\Rightarrow z^2 - 2z + 5$ is a factor of $2z^3 - z^2 + 4z + 15$

$$\begin{array}{r|l} & 2z+3 \\ \hline z^2 - 2z + 5 & 2z^3 - z^2 + 4z + 15 \\ & \underline{2z^3 - 4z^2 + 10z} \\ & 3z^2 - 6z + 15 \\ & \underline{3z^2 - 6z + 15} \\ & 0 \end{array}$$

$$(2z+3)(z^2 - 2z + 15) = 0$$
$$z = -\frac{3}{2} \quad z = 1 + 2i\ and\ z = 1 - 2i$$

Example XI

Given that $2 + 3i$ is a root of the equation $z^3 - 6z^2 + 21z - 26 = 0$. Find the other roots

Solution

$z = 2 + 3i$ is a root $\Rightarrow z = 2 - 3i$ is also a root of the equation $z^3 - 6z^2 + 21z - 26 = 0$

Sum of roots $= 2 + 3i + 2 - 3i$

$= 4$

Product of roots = (2 + 3i)(2 – 3i)

$= 2^2 - (3i)^2$

= 4 + 9

= 13

$\Rightarrow z^2 - 4z + 13$ is a factor of

$z^3 - 6z^2 + 21z - 26 = 0$

$$\begin{array}{r|l} & z-2 \\ \hline z^2 - 4z + 13 & z^3 - 6z^2 + 21z - 26 \\ & \underline{z^3 - 4z^2 + 13z} \\ & -2z^2 + 8z - 26 \\ & \underline{-2z^2 + 8z - 26} \\ & 0 \end{array}$$

$$\Rightarrow (z-2)(z^2 - 4z + 13) = 0$$

$\Rightarrow z = 2, z = 2 + 3i\ and\ z = 2 - 3i$ are roots of equation of $z^3 - 6z^2 + 21z - 26 = 0$

Example XII

Show that $1 + i$ is a root of the equation $z^4 + 3z^2 - 6z + 10 = 0$. Hence find other roots

Solution

$z = 1 + i$

$z^2 = 1 + 2i + i^2$

$z^2 = 1 + 2i - 1$

$z^2 = 2i$

$z^3 = z^2.z$

$= 2i(1+i)$

$= 2i - 2$

$$z^4 = (z^2)^2 = (2i)^2 = 4i^2$$

$= -4$

$\Rightarrow z^4 + 3z^2 - 6z + 10$

$= (-4) + 3(2i) - 6(1+i) + 10$

$= -4 + 6i - 6 - 6i + 10$

$= -10 + 10 + 6i - 6i$

$= 0 + 0i$

$= 0$

$z = 1 + i$ is a root of the equation

$\Rightarrow 1 - i$ is also a root of the equation

Sum of the roots$= 1 + i + 1 - i$

$= 2$

Product of roots = $(1)^2 - i^2$ = 2

$$z^2 - (sum\ of\ roots)z + product = 0$$
$$z^2 - 2z + 2 = 0$$

$\Rightarrow z^2 - 2z + 2$ is a factor of $z^4 + 3z^2 - 6z + 10$

$$\begin{array}{r|l} & z^2+2z+5 \\ \hline z^2-2z+2 & z^4+3z^2-6z+10 \\ & z^4-2z^3+2z^2 \\ \hline & 2z^3+z^2-6z+10 \\ & 2z^3-4z^2+4z \\ \hline & 5z^2-10z+10 \\ & 5z^2-10z+10 \\ \hline & 0 \end{array}$$

$$(z^2-2z+2)(z^2+2z+5)=0$$

$$\Rightarrow z^2+2z+5=0$$

$$z^2-2z+2=0$$

For $z^2 + 2z + 5 = 0$, $z = \dfrac{-2 \pm \sqrt{(2)^2 - 4\times1\times5}}{4\times1}$

$$z=\frac{-2\pm\sqrt{16i^2}}{2}$$

$$z=\frac{-1\pm 4i}{2}$$

$$z=-1+2i$$

$$z=-1-2i$$

$\Rightarrow$ -1 + 2i, -1 – 2i, 1 + i, 1 $-$ i are roots of the equation $z^4+3z^2-6z+10=0$

Example XIII

Show that $1-i$ is a root of the equation $4z^4-8z^3+9z^2-2z+2=0$. Find the other roots.

Solution

$z=1-i$

$z^2=(1-i)^2$

$=1-2i+i^2$

$=-2i$

$z^3=z^2.z$

$=-2i(1-i)$

$=-2i+2i^2$

$=-2-2i$

$z^4=(z^2)^2=(-2i)^2$

$=-4$

$4z^4-8z^3+9z^2-2z+2=$

$4(-4)-8(-2-2i)+9(-2i)-2(1-i)+2$

$=-16+16+16i-18i-2+2i+2$

$=0+0i=0$

Since $z=1-i$ is a of the equation and it implies that 1 + i is also a root.

Sum of roots$=1-i+1+i$

$$=2$$

Product of the roots = $(1+i)(1-i)$

$$1^2-i^2=2$$

$$\Rightarrow z^2-(2z)+2=0$$

$\Rightarrow z^2-2z+2$ is a factor of $z^4-8z^3+9z^2-2z+2=0$.

$$\begin{array}{r|l} & 4z^2+1 \\ \hline z^2-2z+2 & 4z^4-8z^3+9z^2-2z+2 \\ & 4z^4-8z^3+8z^2 \\ \hline & \qquad\qquad z^2-2z+2 \\ & \qquad\qquad z^2-2z+2 \end{array}$$

$$(z^2-2z+1)(4z^2+1)=0$$

$$4z^2=-1 \qquad z^2=-\frac{1}{4}$$

$$\Rightarrow z^2=\frac{1}{4}i^2\ ,\ \ 2=\pm\frac{1}{2}i$$

Example XIV

Given that $z = 2 - i$ is a root of the equation $z^3-3z^2+z+k=0$, k is real. Find other roots.

Solution

$z=2-i$

$z^2=(2-i)^2$

$=4-4i+i^2$

$=3-4i$

$z^3=(2-i)(3-4i)$

$=6-8i-3i+4i^2$

$=2-11i$

$\Rightarrow (2-11i)-3(3-4i)+2-i+k=0$

$2-11i-9+12i+2-i+k=0$

$-11i+11i+4-9+k=0$

$0-5+k=0$

$k=5$

$\Rightarrow z^3-3z^2+z+5=0$

$z=2-i$

$z=2+i$

$$z=2-i$$

$$z=2+i$$

Sum of roots = 4

Product of roots = 5

$z^2-4z+5=0$ is a factor of $z^3-3z^2+z+5=0$

$$\begin{array}{r|l} & z+1 \\ \hline z^2-4z+5 & z^3-3z^2+z+5 \\ & z^3-4z^2+5z \\ \hline & z^2-4z+5 \end{array}$$

$$(z+1)(z^2-4z+5)=0$$

$$(z+1)=0 \quad z=-1$$

$\Rightarrow z=-1, \quad z=2+i, \quad z=2-i$ are roots of the equation $z^3 - 3z^2 + z + k = 0$ where $k = 5$

Example XIV

Solve for z_1 and z_2 in the simultaneous equations below

$$z_1+(1-i)z_2=0$$

$3z_2-3z_1=2-5i$

Solution

$z_1+(1-i)z_2=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$

$3z_2-3z_1=2-5i \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots . (2)$

From eqn (1)

$z_1=-(1-i)z_2$

$substitute\ in\ eqn\ (2)$

$3z_2-3[(-1(1-i)z_2)]=2-5i$

$3z_2+3(1-i)z_2=2-5i$

$3z_2-3iz_2+3z_2=2-5i$

$6z_2-3iz_2=2-5i$

$z_2(6-3i)=2-5i$

$z_2=\dfrac{2-5i}{6-3i}$

$z_2=\dfrac{(2-5i)(6+3i)}{(6-3i)(6+3i)}$

$z_2=\dfrac{12+6i-30i+15}{36+9}$

$z_2=\dfrac{27-24i}{45}$

$z_2=\dfrac{9-8i}{15}$

$z_2=\dfrac{9}{15}-\dfrac{8i}{15}$

$z_1=-(1-i)z_2$

$z_1=-\left((1-i)\left(\dfrac{9-8i}{15}\right)\right)$

$z_1=-\left(\dfrac{9-8i-9i-8}{15}\right)$

$z_1=-\left(\dfrac{1-17i}{15}\right)$

$z_1=\dfrac{-1+17i}{15}$

$z_1=-\dfrac{1}{15}+\dfrac{17i}{15}$

Example XV

Solve the equation $z^3 - 1$

Solution

$$z^3-1=(z)^3-(1)^3$$
$$=(z-1)(z^2+z+1)$$

Since $a^3-b^3=(a-b)(a^2+ab+b^2)$

$$z^3-1=(z-1)(z^2+z+1)=0$$
$$z=1$$
$$z^2+z+1=0$$
$$z=\frac{(-1)\pm\sqrt{(1)^2-4(1)(1)}}{2\times 1}$$
$$z=\frac{-1\pm\sqrt{3i^2}}{2}$$
$$z=-\frac{1}{2}+\frac{(\sqrt{3})i}{2}$$
$$z=-\frac{1}{2}-\frac{\sqrt{3}}{2}i$$
$$z=1, z=-\frac{1}{2}+\frac{\sqrt{3}i}{2}, \qquad z=-\frac{1}{2}-\frac{\sqrt{3}i}{2}$$

Alternatively we can use Demovre's theorem

$$z^3-1=0$$
$$z^3=1$$
$$z^3=1+0i$$
$$z=(1+0i)^{\frac{1}{3}}$$
$$let\ P=1+0i$$
$$|P|=\sqrt{1}=1$$
$$\arg P=tan^{-1}\left(\frac{0}{1}\right)=0$$
$$P=r[\cos(0)+i\sin(0)]$$
$$P=1(\cos 0+i\sin 0)$$
$$z=P^{1/3}$$
$$z=1^{\frac{1}{3}}(\cos(0+360n)+i\sin(0+360n))$$
$$For\ n=0,1,2\ldots$$

(Depending on the number of roots you want)

For $n = 0$, $z=1^{\frac{1}{3}}(\cos(0+360)+i\sin(0+360))^{\frac{1}{3}}$

$$z=1^{\frac{1}{3}}(\cos 120+i\sin 120)$$
$$z=1\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)$$
$$z=-\frac{1}{2}+\frac{\sqrt{3}i}{2}$$

For $n = 1$, $z=1^{\frac{1}{3}}[\cos(0+360\times 1)+i\sin(0+360\times 1)]^{\frac{1}{3}}$

$$z=1(\cos 120+i\sin 120)$$

$$z = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$$

For n = 2, $z = 1^{\frac{1}{3}}[\cos(0 + 360 \times 2) + i\sin(0 + 360 \times 2)]^{\frac{1}{3}}$

$$z = 1(\cos 240 + i \sin 240)$$

$$z = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

$$\Rightarrow z = 1, \qquad z = -\frac{1}{2} + \frac{\sqrt{3}}{2}i, \qquad z = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

Example XVI

Solve: $z^3 + 27 = 0$

Solution

$$z^3 + 3^3 = (z + 3)(z^2 + 3z + 9)$$

From $a^3 + b^3 = (a + b)(a^2 + ab + b^2)$

$$\Rightarrow z^3 + 3^3 = (z + 3)(z^2 + 3z + 9)$$

$$z = -3$$

$$z^2 + 3z + 9 = 0$$

$$z = \frac{-3 \pm \sqrt{3^2 - 4(1)(9)}}{2}$$

$$z = \frac{-3 \pm \sqrt{27i^2}}{2}$$

$$z = -3, \qquad z = -\frac{3}{2} + \frac{3\sqrt{3}i}{2}$$

$$z = \frac{-3}{2} - \frac{3\sqrt{3}}{2}i$$

Alternatively, we can use Demovre's theorem

$$z^3 + 27 = 0$$

$$z^3 = -27$$

$$z = (-27 + 0i)^{\frac{1}{3}}$$

$$let\ P = -27 + 0i$$

$$|P| = \sqrt{(-27)^2 + oi)}$$

$$= 27$$

$$\arg P = 180$$

$$P = 27(\cos 180 + i \sin 180)$$

$$z = P^{\frac{1}{3}} = 27^{\frac{1}{3}}(\cos 180 + i \sin 180)^{\frac{1}{3}}$$

$$z = 27^{\frac{1}{3}}(\cos(180 + 360n) + i \sin(180 + 360n))^{\frac{1}{3}}$$

When n = 0, $z = 27^{\frac{1}{3}}[(\cos 180 + i \sin 180)]^{\frac{1}{3}}$

$$z = 3(\cos 60 + i \sin 60)$$

$$= 3\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)$$

$$= \frac{3}{2} + \frac{3\sqrt{3}}{2}i$$

For n = 1, $z = 27^{\frac{1}{3}}[(\cos(180 + 360 \times 1) + i \sin(180 + 360 \times 1)])^{\frac{1}{3}}$

$$z = 3(\cos 180 + i \sin 180)$$

$$z = -3$$

For n = 2, $z = 27^{\frac{1}{3}}[(\cos(180 + 360 \times 2) + i \sin(180 + 360 \times 2)])^{\frac{1}{3}}$

$$z = 3(\cos 300 + i \sin 300)$$

$$= \frac{3}{2} + -\frac{3\sqrt{3}}{2}i$$

$$= \frac{3}{2} - \frac{3}{2}\sqrt{3}i$$

$z = -3$, $z = \frac{3}{2} + \frac{3\sqrt{3}}{2}i$ and $z = \frac{3}{2} - \frac{3\sqrt{3}}{2}i$

Example XVII

Solve the equation

$z^4 + 1 = 0$

$z^4 = -1 + 0i$

$z^4 = (-1 + 0i)^{\frac{1}{4}}$

$let\ P = -1 + 0i$

$|P| = 1$

$\arg P = 180$

$P = 1(\cos 180 + i \sin 180)$

$$z = P^{\frac{1}{4}} = 1^{\frac{1}{4}}[(\cos(180 + 360n) + i \sin(180 + 360n))]^{\frac{1}{4}}$$

$For\ n = 0,\ z = 1^{\frac{1}{4}}(\cos 45 + i \sin 45)$

$z = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$

For n = 1

$z = 1^{\frac{1}{4}}(\cos 540 + i \sin 540)^{\frac{1}{4}}$

$z = 1(\cos 135 + i \sin 135)$

$z = \frac{-\sqrt{2}}{2} + \frac{i\sqrt{2}}{2}$

For $n = 2$, $z = 1^{\frac{1}{4}}(\cos 900 + i \sin 900)^{\frac{1}{4}}$

$z = 1(\cos 225 + i \sin 225)$

$z = \frac{-\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$

For n = 3

$z = 1^{\frac{1}{4}}(\cos 1260 + i \sin 1260)^{\frac{1}{4}}$

$z = 1(\cos 315 + i \sin 315)$

$z = \left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)$

$For\ z^4 + 1 = 0$

$z = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i,\ \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i,\quad -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i,\ -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$

Example XVIII

Find the fourth roots of -16

Solution

$z = (-16)^{1/4} = (-16+0i)^{1/4}$

Let $P = -16 + 0i$

$$z = P^{\frac{1}{4}} = (-16+01)^{\frac{1}{4}}$$
$$|P| = 16$$
$$\arg P = 180$$
$$z = P^{\frac{1}{4}} = 16^{\frac{1}{4}}[(\cos(180+360n) + i\sin(180+360n))]^{\frac{1}{4}}$$
$$For\ n = 0$$
$$z = 2(\cos 45 + i\sin 45)$$
$$z = \sqrt{2} + \sqrt{2}i$$
$$for\ n = 1$$
$$z = 2(\cos 540 + i\sin 540)$$
$$z = 2(\cos 135 + i\sin 135)$$
$$= -\sqrt{2} + i\sqrt{2}$$

For $n = 2$, $z = 2(\cos 225 + i\sin 225)$
$$= -\sqrt{2} - i\sqrt{2}$$
For $n = 3$, $z = 2(\cos 315 + i\sin 315)$
$$z = \sqrt{2} - \sqrt{2}i$$
$\Rightarrow For\ z = (-16+0i)^{\frac{1}{4}}$
$$z = \sqrt{2} - (\sqrt{2})i, -\sqrt{2} + (\sqrt{2})i$$
$$\sqrt{2} + (\sqrt{2})i, \quad -\sqrt{2} - (\sqrt{2})i$$

Example XIX

Find the cube roots of 27i

$$z = (0+27i)^{\frac{1}{3}}$$
$$let\ P = 0 + 27i$$
$$|P| = \sqrt{0^2 + 27^2}$$
$$= 27$$
$$\arg = tan^{-1}\left(\frac{27}{0}\right) = 90$$
$$P = 27(\cos 90 + i\sin 90)$$
$$z = 27^{\frac{1}{3}}(\cos 90 + i\sin 90)^{\frac{1}{3}}$$
$$z = 27^{\frac{1}{3}}(\cos(90+360n) + i\sin(90+360n))^{\frac{1}{3}}$$
$$For\ n = 0$$
$$z = 3(\cos 30 + i\sin 30)$$
$$z = \frac{3\sqrt{3}}{2} + \frac{3}{2}i$$

$$n = 1$$
$$z = 3(\cos 150 + i\sin 150)$$
$$= 3\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$$
$$= -\frac{3\sqrt{3}}{2} + \frac{3i}{2}$$
$$for\ n = 2$$
$$z = 3(\cos 270 + i\sin 270)$$
$$= -3i$$

Loci in the complex plane

What is a locus

A locus is a path possible position of a variable point, that obeys a given condition. It can be given as Cartesian equation or it can be described in words.

Example I

The complex number z is represented by the point P on the Argand diagram.

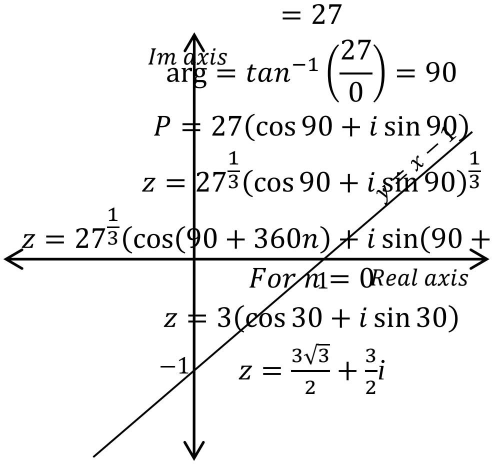
Given that $|z - 1 - i| = |z - 2|$ find in the simplest form the Cartesian equation of the locus

Solution

$$|z - 1 - i| = |z - 2|$$
$$let\ z = x + iy$$

$$|x + iy - 1 - i| = |x + iy - 2|$$
$$|x - 1 + (y-1)i| = |x - 2 + iy|$$
$$\sqrt{(x-1)^2 + (y-1)^2} = \sqrt{(x-2)^2 + y^2}$$
$$(x-1)^2 + (y-1)^2 = (x-2)^2 + y^2$$
$$x^2 - 2x + 1 + y^2 - 2y + 1 = x^2 - 4x + 4 + y^2$$
$$-2x - 2y + 2 = -4x + 4$$
$$2x - 2 = 2y$$
$$y = x - 1$$

The locus is a straight line with a positive gradient $y = x - 1$) which can be represented on the complex plane.



Example II

Given that $|z-2| = 2|z+i|$. Show that the locus of P is a circle.

Solution

$$|z-2| = 2|z+i|$$

Let $z = x + iy$

$$|x+iy-2| = 2|x+iy+i|$$

$$|(x-2)+iy| = 2|x+(y+1)i|$$

$$\sqrt{(x-2)^2+y^2} = 2\sqrt{x^2+(y+1)^2}$$

$$(x-2)^2+y^2 = 4(x^2+(y+1)^2)$$

$$x^2-4x+4+y^2 = 4x^2+4y^2+8y+4$$

$$0 = 3x^2+3y^2+4x+8y$$

$$x^2+y^2+\frac{4}{3}x+\frac{8y}{3} = 0$$

This is sufficient to justify that locus is a circle.

Comparing $x^2+y^2+\frac{4}{3}x+\frac{8y}{3} = 0$ With

$$x^2+y^2+2gx+2fy+c = 0$$

$$2g = \frac{4}{3}$$

$$g = \frac{2}{3}$$

$$2\,fy = \frac{8y}{3}$$

$$f = \frac{4}{3}$$

$$centre(-g,-f)$$

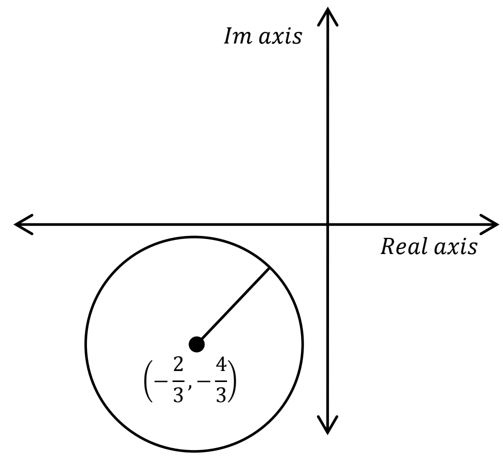
$$centre\left(-\frac{2}{3},-\frac{4}{3}\right)$$

$$r = \sqrt{g^2+f^2-c}$$

$$r = \sqrt{\frac{4}{9}+\frac{16}{9}-0}$$

$$r = \sqrt{\frac{20}{9}}$$

$$r = \frac{2}{3}\sqrt{5}$$



Example III

Show the region represented by $|z-2+i| < 1$

Solution

Let $z = x + iy$

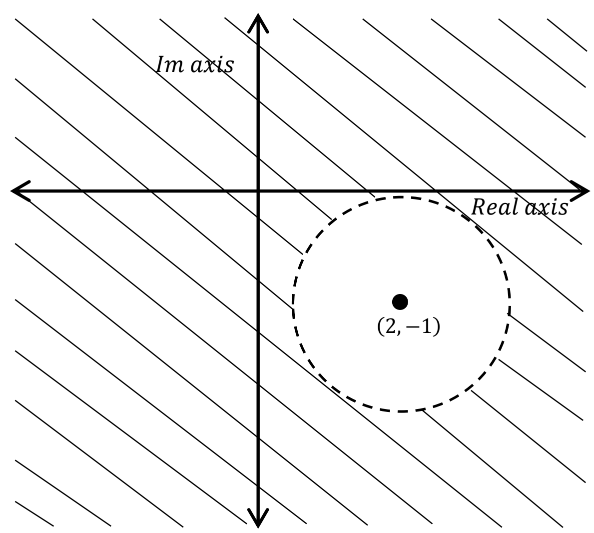
$$|x+iy-2+i|$$

$$|x-2+(y+1)i| < 1$$

$$\sqrt{(x-2)+(y+1)^2} < 1$$

$$(x-2)^2+(y+1)^2 < 1$$

It's a circle with centre (2, -1) and radius less than 1. It can be illustrated on the argand diagram



In order to represent $(x-2)^2+(y+1)^2 < 1$ on the diagram, we can either take a point inside the circle or outside the circle as our test point.

Taking (2,-1) as the test point.

$$\Rightarrow (2-2)^2+(-1+1)^2 < 1$$

$$0 + 0 < 1$$

$$0 < 1$$

$(2, -1)$(the point inside the circle satisfies our locus). It implies that (2,-1) lies in the wanted region. Therefore, we shade the region outside the circle.

Example IV
Given that

$$\left|\frac{z-1}{z+1}\right| = 2$$

find the Cartesian equation of the locus of z and represent the locus by the sketch on the argand diagram. Shade the region for which the inequalities.

$$\left|\frac{z-1}{z+1}\right| > 2$$

Solution

$$z = x + iy$$
$$\left|\frac{x+iy-1}{x+iy+1}\right| = 2$$

$$\left|\frac{(x-1+iy)}{(x+1)+iy}\right| = 2$$

$$\frac{|x-1+iy|}{|(x+1)+iy|} = 2$$

$$|(x-1)+iy| = 2|(x+1)+iy|$$

$$\sqrt{(x-1)^2+y^2} = 2\sqrt{(x+1)^2+y^2}$$

$$(x-1)^2+y^2 = 4((x+1)^2+y^2)$$

$$x^2-2x+1+y^2 = 4(x^2+2x+1+y^2)$$

$$3x^2+3y^2+10x+3 = 0$$
$$x^2+y^2+\frac{10}{3}x+1 = 0$$

The locus is a circle comparing

$$x^{\mathbf{2}}+y^2+\frac{10}{3}x+1 = 0 \text{ with}$$
$$x^2+y^2+2gx+2fy+c = 0$$
$$2g = \frac{10}{3}, \qquad g = \frac{5}{3}, \; 2f = 0 \text{ and } f = 0$$

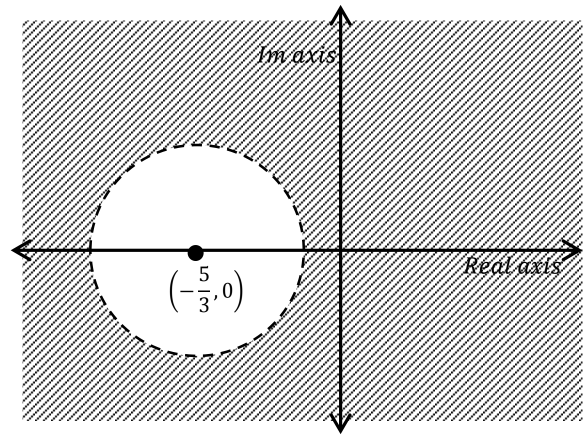
Center $(\frac{-5}{3}, 0)$

$$r = \sqrt{g^2+f^2-c}$$

$$r = \sqrt{\frac{25}{9}+0-1} = \frac{4}{3}$$

For $\left|\frac{z-1}{z+1}\right| > 2$

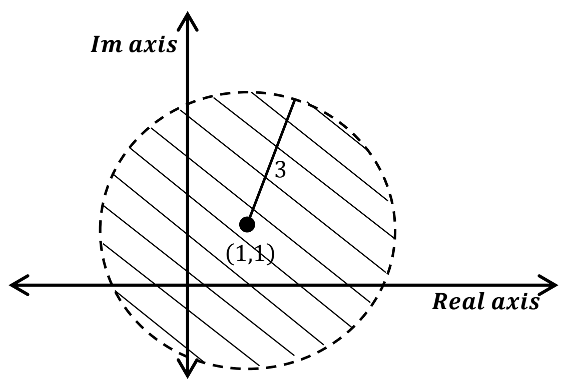
$$\Rightarrow x^2+y^2+\frac{10}{3}x+1 > 0$$



Example V
Shade the region represented by $|z-1-i| < 3$
Solution
Note: Shade the region represented by $|z-1-i| < 3$. Implies that we shade the wanted region.
Let $x+iy$

$$|x+iy-1-i| < 3$$
$$|x-1+i(y-1)| < 3$$
$$\sqrt{(x-1)^2+(y-1)^2} < 3$$
$$(x-1)^2+(y-1)^2 < 9$$

It is a circle with centre (1, 1) and radius less than 9



Taking (1, 1) as our test point

$$(1-1)^2+(1-1)^2$$
$$(0+0) < 9$$

⇒The region inside the circle is the wanted region.

Example VI

Show that when

$$\text{Re}\left(\frac{z+i}{(z+2)}\right) = 0,$$

the point P(x, y) lies on a circle with centre $-1, -\frac{1}{2})$ and radius$\frac{1}{2}\sqrt{5}$

Solution

$$\text{Re}\left(\frac{x+iy+i}{x+iy+2}\right) = 0$$

$$Re\left(\frac{x+(y+1)i}{x+2+iy}\right) = 0$$

$$\text{Re}\left[\frac{(x+(y+1)i)(x+2-iy)}{((x+2)+iy)(x+2-iy)}\right] = 0$$

$$Re\left(\frac{x(x+2)-xyi+(y+1)(x+2)i+y(y+1)}{(x+2)^2+y^2}\right)$$

$$\text{Re}\left(\frac{x^2+2x+y^2+y+[(y+1)(x+2)-xy]i}{(x+2)^2+y^2}\right)$$

$$\text{Re}\left(\frac{x^2+2x+y^2+y}{(x+2)^2+y^2}+\frac{[(y+1)(x+2)-xy]i}{(x+2)^2+y^2}\right) = 0$$

$$\Rightarrow \frac{x^2+2x+y^2+y}{(x+2)^2+y^2} = 0$$

$$x^2+y^2+2x+y = 0$$

Comparing with

$x^2 + y^2 + 2x + y = 0$ with

$$x^2+y^2+2gx+2fy+c = 0$$

$$2g = 2, g = 1$$

$$2fy = y$$

$$f = \frac{1}{2}$$

$$centre\left(-1, -\frac{1}{2}\right)$$

$$radius = \sqrt{g^2+f^2-c}$$

$$= \sqrt{1+\frac{1}{4}-0}$$

$$= \frac{\sqrt{5}}{2}$$

$$= \frac{1}{2}\sqrt{5}$$

Example VII

Given that $z = x + iy$. where x and y are real.

Show that $\text{Im}\left(\frac{z+i}{z+2}\right) = 0$

is equation of a straight line

Solution

$$\text{Im}\left(\frac{x+iy+i}{x+iy+2}\right) = 0$$

$$\text{Im}\left[\frac{(x+(y+1)i)(x+2)-iy}{(x+2+iy)(x+2-iy)}\right] = 0$$

$$Im\left(\frac{x(x+2)-xyi+(y+1)(x+2)i+y(y+1)}{(x+2)^2+y^2}\right)$$

$= 0$

$$\Rightarrow \frac{-xy+(y+1)(x+2)}{(x+2)^2+y^2} = 0$$

$$\frac{-xy+xy+2y+x+2}{(x+2)^2+y^2} = 0$$

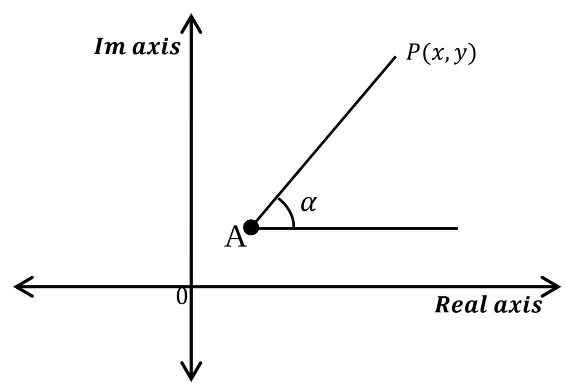
$$2y+x+2 = 0$$

$$y = -\frac{x}{2}+1$$

Which is a straight line with a negative gradient.

Loci in and diagram for arguments of complex numbers

If $\arg(z-A) = \alpha$ is the equation of half line with end point A inclined at an angle α to the real axis



Example I

Sketch the loci defined by the equation

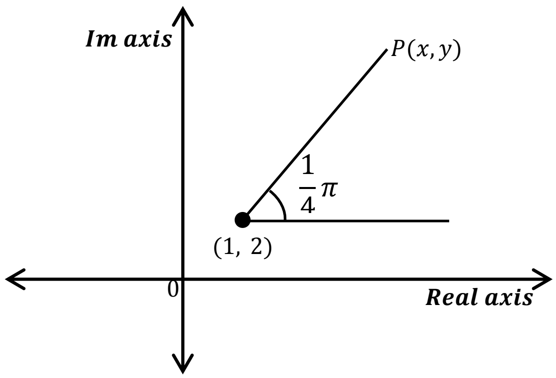
$$\arg(z-1-2i) = \frac{1}{4}\pi$$

Solution

$z - 1 - 2i = z - (1 + 2i)$

Thus if A is a point representing $1 + 2i$

$\arg\big(z - (1 + 2i)\big)$is the angle AP makes with the positive real axis. Hence the equation $\arg(z - 1 - 2i) = \frac{1}{4}\pi$ represents the half line with end point (1, 2) inclined at angle $\frac{1}{4}\pi$ to the real axis.



Example II

Sketch the locus of the equation.

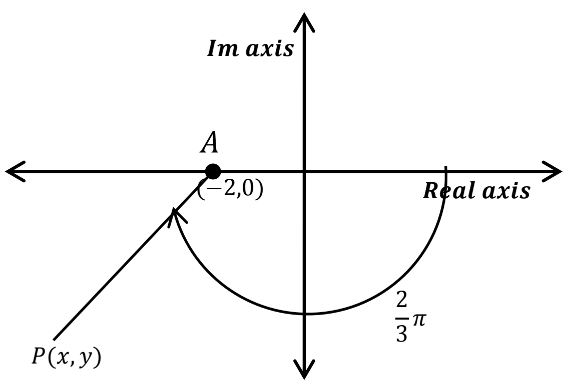
$$\arg(z+2) = -\frac{2}{3}\pi$$

Solution

$$\arg(z+2) = -\frac{2\pi}{3}$$

$$z+2 = (z - \ -2)$$

Thus *A* is a point (-2, 0). Arg(z-2) is the angle AP makes with the real axis. Hence $\arg(z - -2) = -\frac{2}{3}\pi$ represents a half line with end point (-2, 0) inclined at angle $\frac{2}{3}\pi$ measured clockwise from the positive axis.



Example III

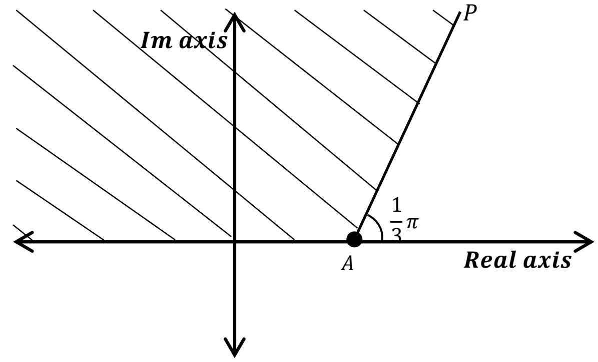
Show by shading the region represented by

$$\frac{1}{3}\pi \leq \arg(z-2) \leq \pi$$

Solution

The equations $\arg(z-2) = \frac{1}{3}\pi$ *and* $\arg(z-2) = \pi$ represent half lines with end point (2, 0). Hence the inequality $\frac{1}{3}\pi \leq \arg(z-2) \leq \pi$

Represent the two lines and region between them



Example IV

Sketch the separate argand diagram the loci defined by

$(i)\ \arg(z+1-3i) = -\frac{1}{6}\pi$

$(ii)\ \arg(z+2+i) = \frac{1}{2}\pi$

Solution

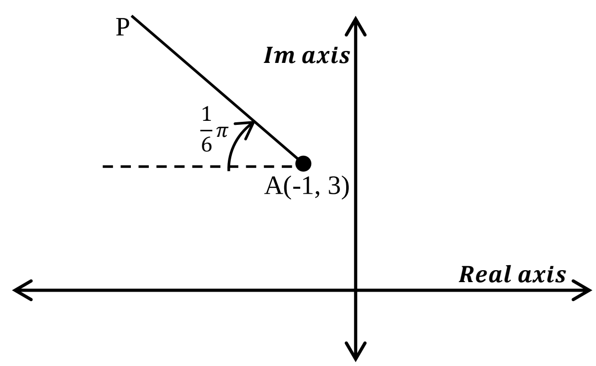
$(\arg(z+1-3i) = -\frac{1}{6}\pi$

$z-(-1+3i) = -\frac{1}{6}\pi$

Thus A is a point (-1, 3)

Arg(z – (-1 + 3*i*) is the angle AP makes with the real axis Hence $\arg(z+1-3i) = -\frac{1}{6}\pi$

is equation of the half line with end point (-1, 3) inclined at an angle of $\frac{1}{6}\pi$ measured clockwise from the real axis

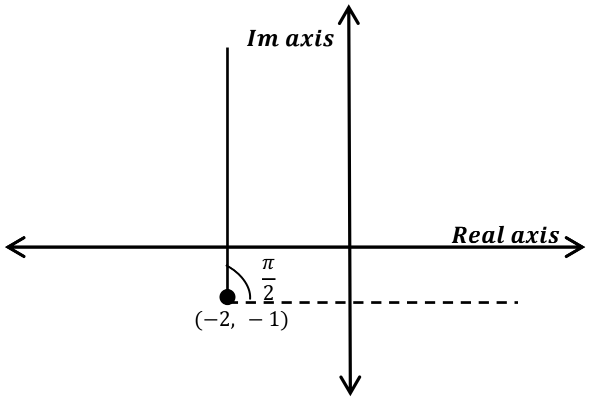


$(ii)\ \arg(z+2+i) = \frac{1}{2}\pi$

$$\arg\big(z-(-2-i)\big) = \frac{1}{2}\pi$$

Thus, point A is $(-2,-1)$.

arg(z $-$(-2 – *i*)) is the angle AP makes with the real axis and $\arg(z+2+i) = \frac{1}{2}\pi$ is the equation of the line through A inclined at and angle of $\frac{1}{2}\pi$ to the real axis



Sketching of loci involving $\boldsymbol{arg}\left(\frac{z-a}{z-b}\right)=\boldsymbol{\gamma}$

Equation involving $\arg\left(\frac{z-a}{z-b}\right)$are more difficult to interprete. If $\arg(z-a)=\alpha$,

$$\arg(z-b)=\beta, \arg\left(\frac{z-a}{z-b}\right)=\gamma,$$

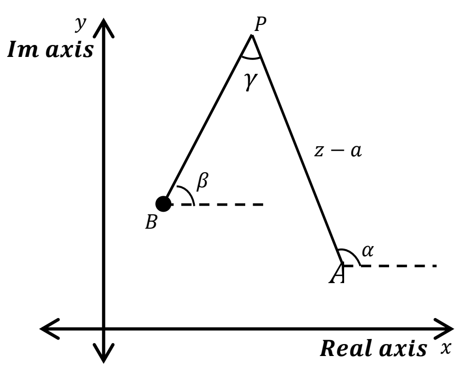
$$\arg(z-a)-\arg(z-b)=\gamma$$

$\alpha-\beta=\gamma.\qquad \gamma=(\alpha-\beta)\pm 2\pi$ if necessary

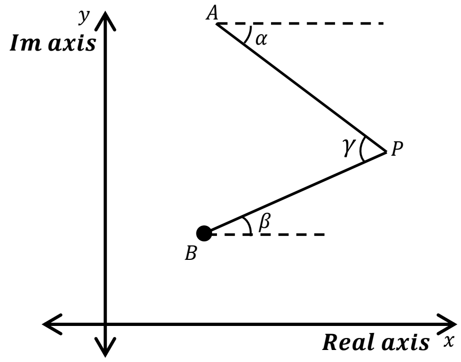
Thus γ is the angle which the vector AP makes with the vector BP.

If the turn from BP to AP is anti-clockwise the α is negative

$$\text{for}\arg\left(\frac{z-a}{z-b}\right)>0$$



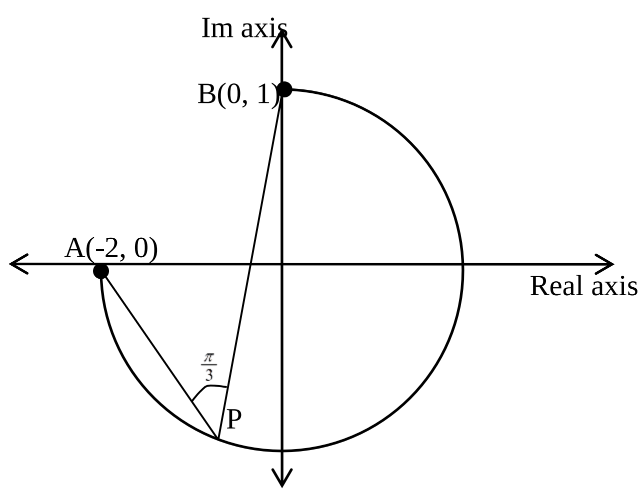
$$\arg\left(\frac{z-a}{z-b}\right)<0$$



For instance, if $\arg\left(\frac{z-3}{z-1}\right)=\frac{1}{4}\pi$, then the locus of P is a circular arc with end point A(3, 0) and (1, 0) such that $\angle APB=\frac{1}{4}\pi$

Similarly if $\arg\left(\frac{z+2}{z-i}\right)=\frac{1}{3}\pi$ then the locus of P is a circular arc with end points A (-2, 0) and B(0, +1) such that $\angle APB=\frac{1}{3}\pi$ since both cases the given arguments are positive, the arcs must be drawn so that the turn from BP to AP is anti-clockwise.

$$\arg\left(\frac{z+2}{z-i}\right)=\frac{1}{3}\pi$$



Example II

Sketch on different argand diagram the loci defined by the equations.

(a) $\arg\left(\frac{z-1}{z+1}\right)=\frac{1}{3}\pi$

(b) $\arg\left(\frac{z-3}{z-2i}\right)=\frac{1}{4}\pi$

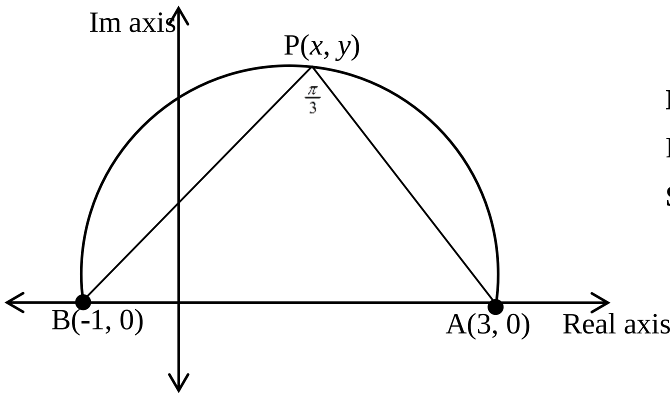
(c) $\arg\left(\frac{z}{z-4+2i}\right)=\frac{1}{2}\pi$

Solution

$$\arg\left(\frac{z-1}{z+1}\right) = \frac{1}{3}\pi$$

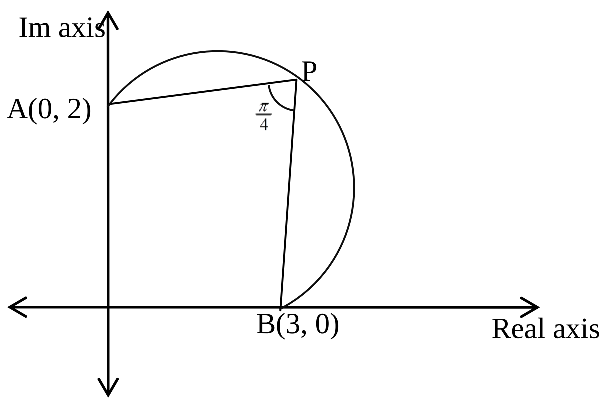
The locus of P is a circular arc with end point A(1, 0) and B(-1, 0) such that

$\angle APB = \frac{1}{3}\pi$



$(b)\ \arg\left(\dfrac{z-3}{z-2i}\right) = \dfrac{1}{4}\pi$

The locus of P is a circular arc with end points (3, 0) (0, 2) such that $\angle APB = \frac{1}{4}\pi$

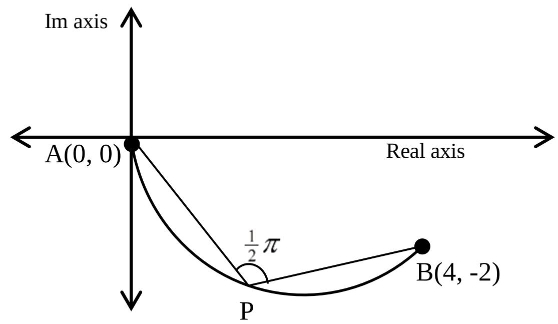


$(c)\ \arg\left(\dfrac{z}{z-4+2i}\right) = \dfrac{1}{2}\pi$

$\arg(\dfrac{z}{z-(4-2i)} = \dfrac{1}{2}\pi$

$\arg\left(\dfrac{z}{z-4+2i}\right)$ is a circle with end points

$A(0,0)$ and $\mathrm{B}(4,-2)$ such that $\angle\mathrm{APB} = \dfrac{1}{2}\pi$



Example

Find the locus of $\arg\left(\frac{z}{z-6}\right) = \frac{\pi}{2}$

Solution

$$let\ z = x + iy$$

$$\arg\left(\frac{z}{z-6}\right) = \arg z - \arg(z-6)$$

$$\Rightarrow\ \arg(z) - \arg(z-6) = \frac{\pi}{2}$$

$$\arg(x+iy) - \arg(x+iy-6) = \frac{\pi}{2}$$

$$\tan^{-1}\left(\tfrac{y}{x}\right) - \tan^{-1}\left(\tfrac{y}{x-6}\right) = \frac{\pi}{2}$$

$$let\ A = tan^{-1}\left(\frac{y}{x}\right)$$

$$\tan A = \frac{y}{x}$$

$$B = \tan^{-1}\left(\frac{y}{x-6}\right)$$

$$\tan B = \frac{y}{x-6}$$

$$(A-B) = \frac{\pi}{2}$$

$$\tan(A-B) = \tan\left(\frac{\pi}{2}\right)$$

$$\frac{\tan A - \tan B}{1 + \tan A + \tan B} = \infty$$

$$\frac{\frac{y}{x} - \frac{y}{x-6}}{1 + \frac{y^2}{x(x-6)}} = \infty$$

$$\frac{\frac{y(x-6)-xy}{x(x-6)}}{\frac{x^2-6x+y^2}{x(x-6)}} = \infty$$

$$\frac{xy - 6y - xy}{x^2 + y^2 - 6^x} = \infty$$

$\Rightarrow x^2 + y^2 - 6x = 0$ which is a circle.

Revision Exercise 1

1. Prove that if $|Z| = \mathbf{r}$, then $ZZ^* = \mathbf{r}^2$.

2. Express $\sqrt{3}$ + i in modulus-argument form. Hence find $(\sqrt{3}+i)^{10}$ and $\frac{1}{(\sqrt{3}+i)^7}$ in the form $a + ib$.
3. Express -1 + i in modulus-argument form. Hence show that $(-1 + i)^{16}$ is real and that $\frac{1}{(-1+i)^6}$ is purely imaginary, giving the value of each.
4. Simplify the following expression:
(a) $\frac{(\cos\frac{2\pi}{7} - i\sin\frac{2\pi}{7})^3}{(\cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7})^4}$ (b) $\frac{(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5})^8}{(\cos\frac{3\pi}{5} - i\sin\frac{3\pi}{5})^3}$
5. Find the expressions for cos 3θ in terms of cos θ,
sin 3θ in terms of sin θ and tan 3θ in terms of tan θ.
6. Express sin 5θ and cos 5θ/cos θ in terms of sin θ.
7. Prove that $\tan 5\theta = \frac{5\tan\theta - 10\tan^3\theta + \tan^5\theta}{1 - 10\tan^2\theta + 5\tan^4\theta}$. By considering the equation tan 5θ = 0, show that $\tan^2(\pi/5) = 5 - 2\sqrt{5}$
8. Find expressions for cos 6θ/sinθ in terms of cos θ and for tan 6θ in terms of tan θ.
9. Express in terms of cosines of multiples of θ:
(a) $\cos^5\theta$ (b) $\cos^7\theta$ (c) $\cos^4\theta$
10. Express in terms of sines of multiples of θ:
(a) $\sin^3\theta$ (b) $\sin^7\theta$ (c) $\cos^4\theta\sin^3\theta$
11. Prove that $\cos^6\theta + \sin^6\theta = \frac{1}{8}(3\cos^4\theta + 5)$
12. Evaluate (a) $\int_0^{\pi} \sin^4\theta \; d\theta$ (b) $\int_0^{\pi/2} \cos^4\theta \sin^2\theta \, d\theta$
13. (a) Express the following complex numbers in a form having a real denominator.
$$\frac{1}{3-2i}, \quad \frac{1}{(1+i)^2}$$
(b) Find the modulus and principal arguments of each of the complex numbers Z = 1 + 2i and W = 2 – I, and represent Z and W clearly by points A and B in an Argand diagram. Find also the sum and product of Z and W and mark the corresponding points C and D in your diagram.
14. If the complex number $x + iy$ is denoted by Z, then the complex conjugate number $x - iy$ is denoted by Z^*,
(a) Express $|Z^*|$ and (Z^*) in terms of $|Z|$ and arg(Z).
(b) If a, b, and c are real numbers, prove that if $aZ^2 + bZ + c = 0$, then then $a(Z^*)^2 + b(Z^*) + c = 0$
(c) If p and q are complex numbers and $q \neq 0$, prove $\left(\frac{p}{q}\right)^* = \frac{p^*}{q^*}$
15. Find the values of a and b such that $(a + ib)^2 = i$. Hence or otherwise solve the equation $z^2 + 2z + 1 - i = 0$, giving your answer in the form $p + iq$, where p and q are real numbers.
16. If $Z = \frac{1}{2}(1+i)$, write down the modulus and argument for each of the numbers Z, Z^2, Z^3, Z^4. Hence or otherwise, show in the Argand diagram, the points representing the number $1 + Z + Z^2 + Z^3 + Z^4$.
17. If $Z = 3 - 4i$, find
(i) Z^* (ii) ZZ^* (iii) $(ZZ)^*$
18. Simplify each of the following:
(a) (3 + 4i) + (2 + 3i) (b) (2 – 4i) – 3(5 – 3i)
(b) $(2i)^2$ (c) i^4
19. Simplify each of the following:
(a) (2 + i)(3 – i) (b) (5 – 2i)(6 + i)
(c) (4 – 3i)(1 – i) (d) (3 + i)(2 – 5i)
20. Express each of the following in the form $a + ib$
(a) $\frac{20}{3+i}$ (b) $\frac{4}{1+i}$
(c) $\frac{2i}{1-i}$ (d) $\frac{1}{1-2i}$
21. Solve the following equations:
(a) $x^2 + 25 = 0$
(b) $2x^2 + 32 = 0$
(c) $4x^2 + 9 = 0$
(d) $x^2 + 2x + 5 = 0$
22. If 3 – 2i and 1 + i are two of the roots of the equation $ax^4 + bx^3 + cx^2 + dx + c = 0$, find the values of a, b, c, d and e.
23. Find the square roots of the following complex numbers:
(a) 5 + 2i
(b) 15 + 8i
(c) 7 – 24i
24. Find the quadratic equations have the roots:
(a) 3i, -3i (b) 1 + 2i, 1 – 2i
(c) 2 + i, 2 – i (d) 2 + 3i, 2 – 3i
25. Find real and imaginary parts of the complex Z when:
(i) $\frac{Z}{Z+1} = 1 + 2i$
(ii) $\frac{Z+i}{Z+1} = \frac{Z+i}{Z-3}$
26. Find the modulus and principal argument of the following complex numbers
(a) 3i (b) 15 (c) -3i (d) -1

27. Find the modulus and principle argument of:

(a) $\frac{1-i}{1+i}$ (b) $\frac{-1-7i}{4+3i}$

(c) $\frac{1+i}{2-i}$ (d) $\frac{(3+i)^2}{1-i}$

28. If Z_1 and Z_2 are complex numbers, solve the simultaneous equations

$4Z^1 + 3Z^2 = 23$

$Z^1 + iZ^2 = 6$

giving your answer in the form $x + iy$

29. Given that $2 + i$ is a root of the equation $Z^3 - 11Z + 20 = 0$. Find the remaining roots.

30. Show that $1 + i$ is a root of the equation $x^4 + 3x^2 - 6x + 10 = 0$. Hence write down the quadratic factor of $x^4 + 3x^2 - 6x + 10$ and find all the roots of the equation.

31. The complex number satisfies the equation $\frac{Z}{Z+2} = 2 - i$. Find the real and imaginary parts of Z and the modulus and argument of Z.

32. If $Z_1 = 4\left(\cos\frac{13\pi}{24} + i\sin\frac{13\pi}{24}\right)$ and $Z_2 = 2\left(\cos\frac{5\pi}{24} + i\sin\frac{5\pi}{24}\right)$, find $\frac{Z_1}{Z_2}$ and Z_1Z_2 in the form $a + ib$.

33. If $Z_1 = 2\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$ and $Z_2 = 6\left(\cos\frac{-3\pi}{4} + i\sin\frac{-3\pi}{4}\right)$, find:

(i) $\left|\frac{Z_1}{Z_2}\right|$ (ii) $\arg\left(\frac{Z_1}{Z_2}\right)$ (iii) $\left|\frac{Z_2}{Z_1}\right|$

(iv) $\arg\left(\frac{Z_2}{Z_1}\right)$

34. One root of the equation $Z^2 + aZ + b = 0$ where a and b are real constants, is $2+3i$. Find the values of a and b.

35. If $Z1$ and $Z2$ are two complex numbers such that $|Z_1 - Z_2| = Z_1 + Z_2|$, show that the difference of their arguments is $\frac{\pi}{2}$ or $\frac{3\pi}{2}$

36. (a) Find the modulus and argument of $\frac{(2-i)^2(3i-1)}{i+3}$

(b) If $Z_1 = \frac{1+7i}{1-i}$ and $Z_2 = \frac{17-7i}{2+2i}$. Find the moduli of Z_1, Z_2, $Z_1 + Z_2$ and Z_1Z_2.

37. Use Demoivre's theorem to show that:

$$\frac{(\cos 3\theta + i\sin 3\theta)^5(\cos\theta - i\sin\theta)^3}{(\cos 5\theta + i\sin 5\theta)^7(\cos 2\theta - i\sin 2\theta)^5} = \cos 13\theta - i\sin 13\theta$$

38. Use Demoivre's theorem to show that:

$\cos 4\theta = \cos^4\theta - 6\cos^2\theta \sin^2\theta + \sin^4\theta$

$\sin 4\theta = 4\cos^3\theta \sin\theta - 4\cos\theta \sin^3\theta$

39. Show that $\left(\frac{1+\sin\theta + i\sin\theta}{1+\sin\theta - i\sin\theta}\right)^n = \cos n(\frac{\pi}{2} - \theta) + i\sin n(\frac{\pi}{2} - \theta)$

40. Use Demoivre's theorem to find the value of $\frac{\sqrt{-3}-1}{\sqrt{-3}+1}$

41. Find the two square roots of I and the four values of $(-16)^{1/4}$.

42. Find the three roots of the equation $(1 - Z)^3 = Z^3$

43. If W is a complex cube root of unity, show that

$$(1 + W - W^2)^3 - (1 - W + W^2)^3 = 0$$

44. Use Demoivre's theorem to find the four fourth roots of $8(-1 + i\sqrt{3})$ in the form $a + ib$, giving a and b correct to 2 decimal places.

45. Use Demoivre's theorem to show that

$$\frac{\cos 5x}{\cos x} = 1 - 12\sin^2 x + 16\sin^4 x$$

46. Prove that if $\frac{Z-6i}{Z+8}$ is real, the locus of the point representing the complex number Z in the Argand diagram is a straight line.

47. Prove that if $\frac{Z-2i}{2Z-1}$ is purely imaginary, the locus of the point representing Z in the Argand diagram is a circle and find its radius.

48. If Z is a complex number and $\left|\frac{Z-i}{Z+1}\right| = 2$, find the equation of the curve in the Argand diagram on which the point representing it lie.

49. The complex numbers $Z - 2$ and $Z - 2i$ have arguments which are

(i) equal and

(ii) differ by $\frac{1}{2}\pi$ and each argument lies between $-\pi$ and π. In each case, find the locus of the point which represents Z in the Argand diagram and illustrate by a sketch.

50. Show by shading on an Argand diagram the region in which both $|Z - 3 - i| \geq |Z - 3 - 5i|$

Answers

1.

2. (a) 1, (b) $-i$ (c) $\frac{1}{2} - \frac{\sqrt{3}}{2}i$ (d) $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$

3. $2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$; $512 - 512\sqrt{3}i$, $\frac{\sqrt{3}}{256} + \frac{1}{256}i$

4. $\sqrt{2}(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4})$; $256 - \frac{1}{8}i$

5. (a) 1, (b) -1

6. $4\cos^3\theta - 3\cos\theta - 4\sin^3\theta$, $\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^3\theta}$

7. $16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$, $1 - 12\sin^2\theta + 16\sin^4\theta$

8. .

9. $32\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1$, $32\cos^5\theta - 32\cos^3\theta + 6\cos = \dfrac{6\tan\theta - 20\tan^3\theta + 6\tan^5\theta}{1 - 15\tan^2\theta + 15\tan^4\theta}$

10. (a) $\frac{1}{16}(\cos 5\theta + 5\cos 3\theta + 10\cos\theta)$,

(b) $\frac{1}{64}(\cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos\theta)$

(c) $\frac{1}{16}(2\cos\theta - \cos 3\theta - \cos 5\theta)$

11. (a) $\frac{1}{4}(3\sin\theta - \sin 3\theta)$, $\frac{1}{64}(35\sin\theta - 21\sin 3\theta + 7\sin\theta - \sin 7\theta)$,

(c) $\frac{1}{64}(3\sin\theta + \sin 3\theta - \sin 5\theta - \sin 7\theta)$

12. (a) $\frac{3\pi}{8}$, (b) $\frac{\pi}{32}$

13. (a) $\dfrac{3+2i}{13}, \dfrac{1}{2}i$ (b) $\sqrt{5}$, 63.4°, $\sqrt{5}$, -26.6°, $3 + i$, $4 + 3i$.

15. $a = \dfrac{1}{\sqrt{2}}, b = \dfrac{1}{\sqrt{2}}$ or $a = \dfrac{-1}{\sqrt{2}}, b = \dfrac{-1}{\sqrt{2}}$

$Z = -1 + \frac{1}{\sqrt{2}+\frac{i}{\sqrt{2}}}$ or $Z = -1 - \frac{1}{\sqrt{2}-\frac{i}{\sqrt{2}}}$

16. $\dfrac{\sqrt{2}}{2}$, 45°; $\dfrac{1}{2}$, 90°; $\dfrac{\sqrt{2}}{4}$, 135°; $\frac{1}{4}$, 180°

17. (i) $3 + 4i$ (ii) 25 (iii) $-7 + 24i$

18. (a) $5 + 7i$ (b) $-13 + 5i$ (c) -4 (d) 1

19 (a) $7 + i$ (b) $32 - 7i$ (c) $1 - 7i$ (d) $11 - 13i$

20. (a) $6 - 2i$ (b) $2 - 2i$ (c) $-1 + i$ (d) $\frac{1}{5} + \frac{2}{5}i$

21 (a) $x \pm 5i$ (b) $x = \pm 4i$ (c) $\pm\frac{3}{2}i$ (d) $x = -1 \pm 2i$

22. $a = 1$, $b = -8$, $c = 27$, $d = -38$, $e = 26$

23. (a) $\pm(3 + 2i)$ (b) $\pm(4 + i)$ (c) $\pm(4 - 3i)$

24. (a) $x^2 + 9 = 0$ (b) $x^2 - 2x + 5 = 0$

(c) $x^2 - 4x + 5 = 0$ (d) $x^2 - 4x + 13 = 0$

25 (i) -1, ½ (ii) $\frac{1}{5}, \frac{-2}{5}$

26. (a) 3, π/2 (b) 15, 0 (c) 3, -π/2 (d) 1, π

27. (a) 1, -π/2 (b) $\sqrt{2}, \frac{-3\pi}{4}$, (c) $\frac{\sqrt{10}}{5}$, 1.25

28. $2 + 3i$ 19. $2 - i$, -4

31. (i) Re(Z) = -3, Im(Z) = -1 (ii) $\sqrt{10}$, -2.82 rads

32. $1 + \sqrt{3}i$; $-4\sqrt{2} + 4\sqrt{2}i$

33. (i) 1/3 (ii) $\frac{-7\pi}{12}$ (iii) 3 (iv) $\frac{7\pi}{12}$

34. -4, 13

36. (a) 5, 0.6435 rad (b) 5, 6.5, 2.061, 32.5

41. $\dfrac{\pm(1+i)}{\sqrt{2}}$, $\pm\sqrt{2} \pm i\sqrt{2}$

42. $\frac{1}{2}, \frac{1}{2}(1 \pm i\sqrt{3})$. 44. $\pm(1.73 + i), \pm(1 - 1.73i)$

47. centre ¼ + i, radius $\frac{1}{4}\sqrt{7}$

48. $\left(x + \dfrac{5}{3}\right)^2 + y^2 = \dfrac{16}{9}$

49. (i) $x + y = 2$ (ii) $(x - 1)^2 + (y - 1)^2 = 2$

Exercise 2

Show on the Argand diagram the region represented by the following:

1. $\arg z = \frac{1}{4}\pi$,
2. $\arg(z - i) = \frac{1}{3}\pi$
3. $\arg(z + 1 - 3i) = \frac{1}{6}\pi$
4. $\arg(z - 3 + 2i) = \pi$
5. $\arg(z + 2 + i) = \frac{1}{2}\pi$
6. $\arg(z - 1 - i) = -\frac{1}{4}\pi$
7. $|z + 1| = |z - 3|$,
8. $|z| = |z - 6i|$
9. $\left|\dfrac{z-i}{z-1}\right| = 1$
10. (a) $\arg\left(\dfrac{z-1}{z+1}\right) = \frac{1}{3}\pi$
11. (a) $\arg\left(\dfrac{z-3}{z-2i}\right) = \frac{1}{4}\pi$ (b) $\left(\dfrac{z}{z-4+2i}\right) = \frac{1}{2}\pi$

In questions 12 to 24 find the Cartesian equation of the locus of the point P representing the complex number z. Sketch the locus of P each case.

12. $2|z + 1| = |z - 2|$
13. $|z + 4i| = 3|z - 4|$
14. $\left|\dfrac{z}{z-4}\right| = 5$
15. $\left|\dfrac{z+i}{z-5-2i}\right| = 1$
16. $\left|\dfrac{z}{z+6}\right| = 5$
17. $\left|\dfrac{z-1}{z+1-i}\right| = \frac{2}{3}$
18. $z - 5 = \lambda i(z + 5)$, where λ is a real parameter
19. $\dfrac{z+2i}{z-2} = \lambda i$, where λ is a real number.
20. $z = 3i + \lambda(2 + 5i)$, where λ is a real parameter.
21. $\mathrm{Im}(z^2) = 2$
22. $\mathrm{Re}\ (z^2) = 1$
23. $\mathrm{Re}\left(z - \dfrac{1}{z}\right) = 0$
24. $\mathrm{Im}\left(z + \dfrac{9}{z}\right) = 0$

In questions 27 to 34 shade in separate Argand diagrams the regions represented by:

25. $|z - i| \le 3$
26. $|z - 4 + 3i| < 4$
27. $0 \le \arg z \le \frac{1}{3}\pi$
28. $\frac{1}{4}\pi < \arg z < \frac{3}{4}\pi$

29. $-\frac{1}{6}\pi < \arg(z-1) < \frac{1}{6}\pi$
30. $-\frac{1}{2}\pi \le \arg(z+i) \le \frac{2}{3}\pi$
31. $|z| > |z + 2|$
32. $|z + i| \le |z - 3i|$
33. Represent each of the following loci in an Argand diagram.
    (a) $\arg(z - 1) = \arg(z + 1)$
    (b) $\arg z = \arg(z - 1$ I $i)$
    (c) $\arg(z - 2) = \pi + \arg z$
    (d) $\arg(z - 1) = \pi + \arg(z - i)$
34. Find the least value of $|z + 4|$ for which
    (a) $\mathrm{Re}(z) = 5$ (b) $\mathrm{Im}(z) = 3$
    (c) $|z| = 1$ (d) $\arg z = \frac{1}{4}\pi$
35. Given that the complex number z varies such that $|z - 7| = 3$, find the greatest and least values of $|z - i|$.
36. Given that the complex number w and z vary subject to the conditions $|z - 12| = 7$ and $|z - i| = 4$, find the greatest and least values of $|w - z|$.
37. In an Argand diagram, the point P represents the complex number z, where $z = x + iy$. Given that $z + 2 = \lambda i(z + 8)$, where λ is a real parameter, find the Cartesian equation of the locus of P as λ varies. If also $z = \mu(4 + 3i)$, where λ is real, prove that there is only one possible position for P.
38. (i) Represent on the same Argand diagram the loci given by the equations $|z - 3| = 3$ and $|z| = |z - 2|$. Obtain the complex numbers corresponding to the point of intersection of these loci. (ii) Find a complex number z whose argument is $\pi/4$ and which satisfies the equation $|z + 2 + i| = |z - 4 + i|$.

Answers

12. $x^2 + y^2 + 4x = 0$, 13. $x^2 + y^2 - 9x - 9x - y + 16$
15. $5x + 3y = 14$. 16. $2x^2 + 2y^2 + 25x + 75 = 0$
17. $5x^2 + 5y^2 - 26x + 8y + 1 = 0$.
18. $x^2 + y^2 = 25$, excluding $(-5, 0)$
19. $x^2 + y^2 - 2x + 2y = 0$, excluding $(2, 0)$.
20. $5x - 2y + 6 = 0$ 21. $xy = 1$. 22. $x^2 - y^2 = 1$
23. $x(x^2 + y^2 - 1) = 0$, excluding $(0, 0)$
24. $y(x^2 + y^2 - 9) = 0$, excluding $(0, 0)$.
34. (a) 9, (b) 3, (c) 3, (d) 4.
35. $5\sqrt{2} + 3$, $5\sqrt{2} - 3$. 38. 24, 2.
37. $x^2 + y^2 + 10x + 16 = 0$
38.(i) $1 \pm i\sqrt{5}$ (ii) $1 + i$.

Revision Exercise 3

Show on the Argand diagram the region represented by the following:

1. $\left(\dfrac{z+1}{z-1}\right) = \frac{1}{3}\pi$
2. $\left|\dfrac{z-2-3i}{z+2+i}\right| = 1$
3. Express the complex number $z_1 = \dfrac{11+2i}{3-4i}$ in the form $x + iy$ where x and y are real. Given that $z_2 = 2 - 5i$, find the distance between the points in the Argand diagram which represent z_1 and z_2. Determine the real numbers α and β such that $\alpha z_1 + \beta z_2 = -4 + i$.
4. (i) Find two complex numbers z satisfying the equation $z^2 = -8 - 6i$.
   (ii) Solve the equation $z^2 - (3 - i)z + 4 = 0$ and represent the solutions on an Argand diagram by vectors $\overrightarrow{OA}$ and $\overrightarrow{OB}$, where **O** is the origin. Show that triangle OAB is right-angled.
5. If z and w are complex numbers, show that:
   $$|z-w|^2 + |z+w|^2 = 2\{|z|^2 + |w|^2\}$$
   Interpret your results geometrically.
6. A regular octagon is inscribed in the circle $|z| = 1$ in the complex plane and one of its vertices represents the number $\dfrac{1}{\sqrt{2}}(1+i)$. Find the numbers represented by the other vertices.
7. (i) Two complex numbers z_1 and z_2 each have arguments between 0 and π. If $z_1z_2 = i - \sqrt{3}$ and $\dfrac{z_1}{z_2} = 2i$, find the values of z_1 and z_2 giving the modulus and argument of each.
   (ii) Obtain in the form $a + ib$ the solutions of the equation $z^2 - 2z + 5 = 0$, and represent the solutions on an Argand diagram by the points A and B.
   The equation $z^2 - 2pz + q = 0$ is such that p and q are real, and its solutions in the Argand diagram are represented by the points C and D. Find in the simplest form the algebraic relation satisfied by p and q in each of the following cases:
   (a) $p^2 < q$, $p \neq 1$ and A, B, C, D are the vertices of a triangle;
   (b) $p^2 > q$ and $\angle CAD = \frac{1}{2}\pi$
8. (a) If $-\pi < \arg z_1 + \arg z_2 \le \pi$, show that $\arg(z_1z_2) = \arg z_1 + \arg z_2$. The complex numbers $a = 4\sqrt{3} + 2i$ and $b = \sqrt{3} + 7i$ are represented in the Argand diagram by points A and B respectively. O is the origin. Show that triangle OAB is equilateral and find the complex number c which the point C represents where $OABC$ is a rhombus. Calculate $|c|$ and $\arg c$.

(b) z is a complex number such that $z = \dfrac{p}{2-q} + \dfrac{q}{1+3i}$ where p and q are real. If $\arg z = \pi/2$ and $|z| = 7$ find the values of p and q.

9. .

10. (a) Show that $(1 + 3i)^3 = -(26 + 18i)$.
(b) Find the three roots z_1, z_2, z_3 of the equation $z^3 = -1$
(c) Find in the form $a + ib$, the three roots z'_1, z'_2, z'_3 of the equation $z^3 = 26 + 18i$.
(d) Indicate in the same Argand diagram the points represented by z_r and z'_r for $r = 1, 2, 3$, and prove that the roots of the equations may be paired so that $|z_1 - z_2| = z_2 - z'_2 = |z_3 - z'_3| = 3$.

11. Write down or obtain the non-real cube roots of unity, w_1 and w_2, in the form $a + ib$, where a and b are real. A regular hexagon is drawn in an Argand diagram such that two adjacent vertices represent w_1 and w_2, respectively and centre of the circumscribing circle of the hexagon is the point (1, 0). Determine in the form $a + ib$, the complex numbers represented by the other four vertices of the hexagon and find the product of these four complex numbers.

12. A complex number w is such that $w^3 = 1$ and $w \neq 1$. Show that:
(i) $w^2 + w + 1 = 0$
(ii) $(x + a + b)(x + wa + w^2b)(x + w^2a + wb)$
is real for real x, a and b, and simplify this product. Hence or otherwise find the three roots of the equation $x^3 - 6x + 6 = 0$, giving your answers in terms of w and cube roots of integers.

13. (i) Find, without the use of tables, the two square roots of $5 - 12i$ in the form $x + iy$, where x and y are real.
(ii) Represent on an Argand diagram the loci $|z - 2| = 2$ and $|z - 4| = 7$. Calculate the complex numbers corresponding to the points of intersection of these loci.

14. (i) Given that $(1 + 5i)p - 2q = 7i$, find p and q when (a) p and q are real (b) p and q are conjugate complex numbers.
(ii) Shade on the Argand diagram the region for which $3\pi/4 < \arg z < \pi$ and $0 < |z| < 1$. Choose a point in the region and label it A. If A represents the complex number z, label clearly the points B, C, D and E which represent $-z$, iz, $z + 1$ and z^2 respectively.

15. (i) Show that $z = 1 + i$ is a root of the equation $z^4 + 3z^2 - 6z + 10 = 0$. Find the other roots of the equation.
(ii) Sketch the curve in the Argand diagram defined by $|z - 1| = 1$, $\text{Im } z \geq 0$. Find the value of z at the point P in which this curve is cut by the line $|z - 1| = |z - 2|$. Find also the value of $\arg z$ and $\arg(z - 2)$ at P.

16. (i) If $z = 1 + i\sqrt{3}$, find $|z|$ and $|z^5|$, and also the values of $\arg z$ and $\arg(z^5)$ lying between $-\pi$ and π. Show that $\text{Re}(z^5) = 16$ and find the value of $\text{Im}(z^5)$.
(ii) Draw the line $|z| = |z - 4|$ and the half line $\arg(z - i) = \pi/4$ in the Argand diagram. Hence find the complex number that satisfies both equations.

17. (i) Without using tables, simplify $\dfrac{(\cos\frac{\pi}{9} + i\cos\frac{\pi}{9})^4}{(\cos\frac{\pi}{9} - i\sin\frac{\pi}{9})^5}$.
(ii) Express $z_1 = \dfrac{7+4i}{3-2i}$ in the form $p + qi$, where p and q are real. Sketch in an Argand diagram the locus of the points representing complex numbers z such that $|z - z_1| = \sqrt{5}$. Find the greatest value of z subject to this condition.

18. (i) Given that $z = 1 - i$, find the values of $r(>0)$ and θ, $-\pi < \theta < \pi$, such that $z = r(\cos\theta + i\sin\theta)$. Hence or otherwise find $1/z$ and z^6, expressing your answers in the form $p + iq$, where $q, r \in \mathbb{R}$.
(ii) Sketch on an Argand diagram the set of points corresponding to the set A, where $A = \{z : z \in \mathbb{C}, \arg(z - i) = \pi/4\}$. Show that the set of points corresponding to the set B, where $B = \{z : z \in \mathbb{C}, |z + 7i| = 2|z - 1|\}$, forms a circle in the Argand diagram. If the centre of this circle represents the numbers z_1, show that $z_1 \in A$.

19. Use De Moivre's theorem to show that
$$\cos 7\theta = 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta$$

20. (i) If $(1 + 3i)z_1 = 5(1 + i)$, express z_1 and z_1^2 in the form $x + iy$, where x and y are real. Sketch in an Argand diagram the circle $|z - z_1| = |z_1|$ giving the coordinates of its centre.
(ii) If $z = \cos\theta + i\sin\theta$, show that:
$$z = \frac{1}{z} = 2i\sin\theta \qquad z^n = \frac{1}{z^n} = 2i\sin n\theta$$
Hence or otherwise, show that
$$16\sin^5\theta = \sin 5\theta - 5\sin 3\theta + 10\sin\theta$$

21. .

22. (i) Given that x and y are real, find the values of x and y which make satisfy the equation $\dfrac{2y+4i}{2x+y} - \dfrac{y}{x-i} = 0$

(ii) Given that $z = x + iy$, where x and y are real, (a) Show that $\operatorname{Im}\left(\frac{z+i}{z+2}\right) = 0$, the point (x, y) lies on a straight line (b) Show that, when $\operatorname{Re}\left(\frac{z+i}{z+2}\right) = 0$, the point (x, y) lies on a circle with centre (-1, -½) and radius $\frac{1}{2}\sqrt{5}$

23. (i) Find $|z|$ and arg z for which the complex numbers z given by (a) $12 - 5i$, (b) $\frac{1+2i}{2-i}$, giving the argument in degrees (to the nearest degree) such that $-180° < \arg z \le 180°$.

(ii) By expressing $\sqrt{3} - i$ in modulus-argument form, or otherwise, find the least positive integer n such that $(\sqrt{3} - i)^n$ is real and positive.

(iii) The point P in the Argand diagram lies outside or on the circle of radius 4 with centre at (-1, -1). Write down in modulus form the condition satisfied by the complex number z represented by point P.

24. Sketch the circle C with Cartesian equation $x^2 + (y - 1)^2 = 1$. The point P representing the non-zero complex number z lies on C. Express $|z|$ in terms of θ, the argument of z. Given that $z' = 1/z$, find the modulus and argument of z' in terms of θ. Show that, whatever the position of P on the circle C, the point P' representing z' lies on a certain line, the equation of which is to be determined.

25. (a) The sum of the infinite series $1 + z + z^2 + z^3 + \ldots$ for values of z such that $|z| < 1$ is $1/(1 - z)$. By substituting $z = ½(\cos\theta + i\sin\theta)$ in this result and using De Moivre's theorem, or otherwise, prove that

$$\frac{1}{2}\sin\theta + \frac{1}{2^2}\sin 2\theta + \frac{1}{2^n}\sin n\theta + \ldots = \frac{2\sin\theta}{5 - 4\cos\theta}$$

Answers

3. $1 + 2i$; $5\sqrt{2}$; -2, -1
4. (i) $\pm(1 - 3i)$, (ii) $2 - 2i$, $1 + i$
5. sum of squares of a parallelogram = sum of squares of sides
6. ± 1, $\pm i$, $\pm\frac{1}{\sqrt{2}}(1 - i)$, $\frac{1}{\sqrt{2}}(1 + i)$
7. (i) $-1 + i\sqrt{3}$, 2, $2\pi/3$; $\frac{\sqrt{3}}{2} + \frac{1}{2}i$, 1, $\pi/6$

   (ii) $1 \pm 2i$; (a) $p^2 = q - 4$, (b) $2p = q + 5$
8. (a) $-3\sqrt{3} + 5i$; $2\sqrt{13}$, 2.38 rad, (b) $5, -20$.
9. (i) $-1 - i$, $3\pi/4$, (ii) $2 - i$, 2; -10.
10. (b) -1, $\frac{1}{2} \pm \frac{3}{2}i$, (c) $-1 - 3i$, $\frac{1}{2}(1 - 3\sqrt{3}) + \frac{1}{2}(3 + \sqrt{3})i$
11. $-\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$; $1 \pm i\sqrt{3}$;, $\frac{5}{2} \pm \frac{\sqrt{3}}{2}i$; 28
12. (ii) $x^3 - 3abx + a^3 + b^3$; $\sqrt[3]{2} - \sqrt[3]{4}$, $\omega\sqrt[3]{2} - \omega^2\sqrt[3]{4}$, $\omega^2\sqrt[3]{2} - \omega\sqrt[3]{4}$
13. (i) $\pm(3 - 2i)$, (ii) $3 \pm i\sqrt{3}$.
14. (i) (a) 7/5, -4/5; (b) $2 \pm i$
15. (i) $1 - i$, $-1 \pm 2i$, (ii) $\frac{1}{2}(3 + i\sqrt{3})$; $\pi/6$, $2\pi/3$
16. (i) 2, 32, $\pi/3$, $-\pi/3$; $-16\sqrt{3}$, (ii) $2 + 3i$
17. (i) -1, (ii) $1 + 2i$, $2\sqrt{5}$
18. (i) $\sqrt{2}$, $-\pi/4$; $\frac{1}{2} + \frac{1}{2}i$, $8i$.
20. (i) $2 - i$, $3 - 4i$; (2, -1)
21. (ii) $3x^2 + 3y^2 + 10x + 3 = 0$.
22. (i) $x = 1$, $y = 2$ or $x = -1$, $y = -2$
23. (i) a) 13, -23°, (b) 1, 90°; (ii) 12;
    (iii) $|z + 1 + i| \ge 4$
24. $2\sin\theta$; $\frac{1}{2}\operatorname{cosec}\theta$, $-\theta$; $y = -\frac{1}{2}$. 25. $2\sqrt{2} - 2$.